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A fresh look at midpoint singularities in the algebra of string fields

Theodore G. Erler

University of California, Santa Barbara
Santa Barbara, CA 93106, U.S.A
*E-mail:*terler@physics.ucsb.edu

Abstract: In this paper we study the midpoint structure of the algebra of open strings using its formulation as a continuous Moyal product as described in [1]. We explain how poles in the open string non(anti)commutativity parameter correspond to certain "null" operators which annihilate the vertex, implying that states proportional to such operators tend to have vanishing star product with other string fields. We also identify in infinite sequence of new (anti)commutative coordinates and null operators for the star product in in the complex plane. These properties can be thought of as characteristic of the fact that the string is a continuous object and cannot be broken into "halves" as the split string intuition suggests. In fact, we show that the above properties can be anomalous if we invoke half-string degrees of freedom which cannot be represented with full string Fourier modes. The most prominent example occurs when multiplying a null operator with the identity string field.

Keywords: String Field Theory, Non-Commutative Geometry.

JHEP03(2005)042

Contents

1. Introduction

Cubic string field theory began with a simple insight, due to Witten [2]: If a string field is imagined as a matrix whose indices correspond to the left and right halves of the string, matrix multiplication generates an associative algebra suitable for describing string interactions. This is a crucial insight, but Witten himself was the first to warn that it should be taken with a grain of salt. A string field is really a functional of a full and continuous string; to make it a matrix we would have to split the string into halves, which brings up all sorts of difficult questions, for example how to set boundary conditions at the midpoint, what the left/right splitting does to the kinetic term, midpoint anomalies, etc. Yet, if Witten's product is *not* a local matrix product, what precisely *is it*? The answer remains unclear. Usually, the three string interaction vertex $\langle V_3|$ is taken as a working "definition" of the product. Though it was initially constructed [3] using matrix-like half string overlaps, once the precise form of $\langle V_3|$ had been determined it seemed reasonable to leave the heuristic picture of "matrix multiplication" to the side and work with concrete formulas. Nevertheless, with the discovery of associativity anomalies [4] and related problems in the context of the purely cubic string field theory [5], it became clear that, even with explicit formulas at hand, the singular nature of Witten's product could cause serious problems with any attempt to understand the theory analytically.

More recently, work in string field theory has been motivated by finding evidence, numerically or analytically, for Sen's well-known conjectures [6] regarding tachyon condensation. From the analytic side, the vacuum string field theory (VSFT) model [7, 8] —

intended to describe physics around the locally stable closed string vacuum — has been particularly instructive. But because of the trivial form of the propagator, the interaction plays a defining role, and again questions regarding the singular nature of Witten's product are of central importance [7]–[11].

In this paper we investigate the open string star product in an attempt to understand how the naive matrix product picture is modified at the string midpoint. We consider Witten's product in the simplest case of zero momentum and in Siegel gauge [13], where we will use of the fact that the product can be expressed as a continuous composition of Moyal products [1, 12], labeled by a parameter kappa $= \kappa \geq 0$, with a characteristically varying non(anti)commutativity parameter $\theta(\kappa) = \tanh\frac{\pi\kappa}{4}$ for the matter sector and $g(\kappa) = \coth\frac{\pi\kappa}{4}$ for the ghost sector. Our results can be summarized as follows. The breakdown of the matrix-like picture of the star product comes from the behavior of certain key midpoint degrees of freedom under multiplication. As is already known [1], the matter sector contains a coordinate P_L, describing the momentum of the left half of a string, which commutes with the action of the star product:

$$P_L(\Psi * \Phi) = (P_L\Psi) * \Phi = \Psi * (P_L\Phi). \tag{1.1}$$

In the ghost sector we discover a pair of midpoint coordinates which, surprisingly, *annihilate* the star product — they are the b-ghost momentum at the string midpoint, $\pi_b(\frac{\pi}{2})$, and the c-ghost momentum of the left half of the string, $\Pi_{c,L}$, satisfying:

$$\begin{aligned} \Pi_{c,L}(\Psi *_{b_0} \Phi) &= (\Pi_{c,L}\Psi) *_{b_0} \Phi = \Psi *_{b_0} (\Pi_{c,L}\Phi) = 0 \\ \pi_b\left(\frac{\pi}{2}\right)[\Psi *_{b_0} \Phi] &= \left[\pi_b\left(\frac{\pi}{2}\right)\Psi\right] *_{b_0} \Phi = \Psi *_{b_0} \left[\pi_b\left(\frac{\pi}{2}\right)\Phi\right] = 0\,, \end{aligned} \tag{1.2}$$

where $*_{b_0}$ is the "reduced" star product in Siegel gauge [13]. In addition to the above mentioned coordinates, we discover an infinite sequence of new commutative and null coordinates for both the matter and ghost sectors, labeled in a distributional sense by complex κ and corresponding to ever higher (worldsheet) derivatives of string degrees of freedom at the midpoint. These properties directly contradict the interpretation of the star product as a matrix product — the only commutative elements of a matrix algebra are pure numbers, and, because a matrix algebra has an identity, it can have no null elements. However, equations (1.1), (1.2) can be anomalous if the fields Φ, Ψ being multiplied are not sufficently well behaved. As we will see in a few examples, this seems to happen when multiplication requires us to invoke naively unphysical degrees of freedom allowing the string to "break in half," so the matrix product picture is applicable. A prominent example of such an unphysical coordinate describes a string whose midpoint disconnects from the rest of the string.

This paper is organized as follows. In section 2 we review the Moyal formulation of the star product in the matter sector together with issues associated with the commutative nature of P_L. Section 3 explains the pitfalls of attempting to formulate the star product as a matrix product via the split string formalism in the κ basis. In section 4 we consider the ghost sector, which appears to be more singular than the matter sector due to the appearance of a *pole* in the non-anticommutativity parameter $g(\kappa)$. We find that this pole implies the existence of null operators satisfying eq.1.2. In section 5 we attempt to reconcile null operators with the identity string field, demonstrating the existence of anomalies in expressions involving both the identity string field and null operators. In section 6 we consider the possibility of Moyal coordinates labelled (in a distributional sense) by complex kappa. This allows us to discover an infinite sequence of new (anti)commutative coordinates and null operators for the open string star product in both the matter and ghost sectors. We end with a few concluding remarks.

2. Review of the Moyal formalism in the matter sector

A classical open string field $|\Psi\rangle$ is an element of the state space of a particular matter-ghost boundary conformal field theory(BCFT):

$$|\Psi\rangle \in \mathcal{H}_{\rm BCFT} = \mathcal{H}_{\rm matter} \otimes \mathcal{H}_{\rm ghost}\,.$$

Typically, we are interested in the BCFT corresponding to an open string living on a space-filling D-25 brane, in which case we have Neumann boundary conditions in all directions at the string endpoints. Let us focus on the matter sector of the state space. We define "position" and "momentum" operators on the worldsheet in terms of the standard mode oscillators α_n:

$$\begin{aligned} x(\sigma) &= x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma \qquad & x_n &= \frac{i}{n\sqrt{2}}(\alpha_n - \alpha_{-n}) \\ \pi p(\sigma) &= p + \sqrt{2}\sum_{n=1}^{\infty} p_n \cos n\sigma \qquad & p_n &= \frac{1}{\sqrt{2}}(\alpha_n + \alpha_{-n})\,, \end{aligned}$$

where $[\alpha_m, \alpha_n] = m\delta_{m,-n}$ (suppressing Lorentz indices). The modes x_n and p_n have eigenstates:

$$\begin{aligned} \langle x_n| &= \langle 0|\mathcal{O}(x_n)\,, \qquad & \mathcal{O}(x_n) &= \exp\left[-\frac{n}{2}(x_n^2 - 2\sqrt{2}i\alpha_n x_n - \alpha_n^2)\right] \\ \langle p_n| &= \langle 0|\tilde{\mathcal{O}}(p_n)\,, \qquad & \tilde{\mathcal{O}}(p_n) &= \exp\left[-\frac{1}{2n}(p_n^2 - 2\sqrt{2}\alpha_n p_n + \alpha_n^2)\right]\,, \end{aligned}$$

where x_n, p_n now refers to the eigenvalue and $|0\rangle$ is the (matter sector) $SL(2,\mathbb{C})$ vacuum. In order to formulate the open string star product as a Moyal product, we will find it necessary to represent the string field as a functional of the *even* position Fourier modes and the *odd* momentum Fourier modes (restricting ourselves to zero momentum[1]):

$$\Psi[x_{2n}, p_{2n-1}] = \langle 0|\prod_{n=1}^{\infty}\mathcal{O}(x_{2n})\tilde{\mathcal{O}}(p_{2n-1})|\Psi\rangle\,. \tag{2.1}$$

It is useful to think of the modes x_{2n}, p_{2n-1} as defining a choice of coordinates (sometimes called the "mixed basis") on a linear space $\mathbb{P}$. A string field is then simply complex valued

[1] The Moyal formulation of the star product at nonzero momentum seems complicated [14] and is not as well understood. We will not directly address these issues in this paper.

JHEP03(2005)042

functional on $\mathbb{P}$. Vectors in $\mathbb{P}$ have a natural inner product:

$$\langle u, u' \rangle = \sum_{n=1}^{\infty} \left[\frac{1}{2n-1} p_{2n-1} p'_{2n-1} + 2n x_{2n} x'_{2n} \right], \qquad u, u' \in \mathbb{P}$$

induced by considering string configurations on which fields in the perturbative Fock space have nonvanishing support.[2]

The open string star product of functionals on $\mathbb{P}$ can be calculated with the formula,

$$\Psi * \Phi(u) = \mathcal{N} \; \Psi \star \Phi(u) \,,$$

where $\mathcal{N}$ is a normalization[3] and $\star$ is a canonically normalized Moyal product satisfying,

$$[x_{2n}, p_{2m-1}]_\star = 2i \; T_{2n,2m-1} \,, \tag{2.2}$$

with the other Moyal commutators vanishing. The matrix T above is given by ,

$$\begin{aligned} T_{2n,2m-1} &= \frac{4}{\pi} \int_0^{\pi/2} d\sigma \cos 2n\sigma \cos(2n-1)\sigma \\ &= \frac{2(-1)^{m+n+1}}{\pi} \left(\frac{1}{2m-1+2n} + \frac{1}{2m-1-2n} \right). \end{aligned} \tag{2.3}$$

The string modes x_{2n}, p_{2n-1} are only a particular basis for $\mathbb{P}$. Clearly, it is worthwhile to search for another basis in which the noncommutativity parameter takes a simpler form. We will have more to say about this in a moment.

It is worthwhile saying a few more words about the linear operator T. We can think of T as a map between two Hilbert spaces $\mathcal{H}_{odd}$ and $\mathcal{H}_{even}$:

$$T : \mathcal{H}_{\text{odd}} \to \mathcal{H}_{\text{even}} \,.$$

These two Hilbert spaces can be characterized by introducing a basis of "states,"

$$|2n-1\rangle \in \mathcal{H}_{\text{odd}} \qquad |2n\rangle \in \mathcal{H}_{\text{even}} \,, \qquad n \geq 1 \tag{2.4}$$

so that,

$$\begin{aligned} x_{2n} &\equiv \langle 2n | x \rangle \qquad p_{2n-1} \equiv \langle 2n-1 | p \rangle \\ T_{2m,2n-1} &= \langle 2m | T | 2n-1 \rangle \,. \end{aligned} \tag{2.5}$$

If we define the inner product on $\mathbb{P}$ to be,

$$\langle u, u' \rangle = \langle x | x' \rangle + \langle p | p' \rangle$$

[2]Specifically, the vacuum state takes the form $\Psi_{|0\rangle}(u) = \exp(-\frac{1}{2}\langle u, u \rangle)$.

[3]This normalization constant appears to be infinite, even in $D = 26$ when ghosts are taken into account [15]. It's not clear how this divergence should be physically interpreted or regulated. Our guess is that something to do with the kinetic operator L_0, which is not a "bounded" operator and appears somewhat singular in the continuous basis; perhaps this infinite constant is necessary to compensate for this in the interaction term.

JHEP03(2005)042

we must normalize the basis as

$$\langle 2m|2n\rangle = \frac{1}{2m}\delta_{2m,2n} \qquad \langle 2m-1|2n-1\rangle = (2m-1)\delta_{2m-1,2n-1}\,.$$

Since $|2n\rangle, |2n-1\rangle$ are not normalized to one, in this basis $\mathcal{H}_{\rm odd}, \mathcal{H}_{\rm even}$ are $-\frac{1}{2}, \frac{1}{2}$ order Sobelev spaces respectively. With these definitions, T is a well-defined, bounded linear operator between $\mathcal{H}_{\rm odd}$ and $\mathcal{H}_{\rm even}$. In fact, T has an inverse [16, 10], an operator usually called R in the literature:

$$R : T(\mathcal{H}_{\rm odd}) \subset \mathcal{H}_{\rm even} \to \mathcal{H}_{\rm odd}\,, \qquad RT = 1_{\rm odd}\,, \qquad TR = 1_{\rm even}\,. \tag{2.6}$$

with,[4]

$$\begin{aligned} R_{2m-1,2n} = \frac{2n}{2m-1}\langle 2n-1|R|2m\rangle &= \frac{4}{\pi}\int_0^{\pi/2} d\sigma\cos(2m-1)\sigma[\cos 2n\sigma - (-1)^n] \\ &= \frac{4n(-1)^{n+m}}{\pi(2m-1)}\left(\frac{1}{2m-1+2n} - \frac{1}{2m-1-2n}\right) \\ &= \frac{(2n)^2}{(2n-1)^2}T_{2n,2m-1}\,. \end{aligned} \tag{2.7}$$

Unfortunately, R is more singular than T — it is not a bounded linear operator on all of $\mathcal{H}_{\rm even}$.[5] This is related to the fact that T has a zero-mode [10]:

$$T|v\rangle = 0\,, \qquad T_{2n,2m-1}v_{2m-1} = 0\,,$$

where following ref. [10] we have defined,

$$v_{2n-1} = \frac{2\sqrt{2}}{\pi}\frac{(-1)^{n+1}}{2n-1} = \frac{1}{2n-1}\langle 2n-1|v\rangle \tag{2.8}$$

$|v\rangle$ is not normalizable[6] in $\mathcal{H}_{\rm odd}$, consistent with the fact that T is invertible there. However, given the fact that $RT = 1$ in $\mathcal{H}_{\rm odd}$ it is natural to define $RT = 1$ by extension on the whole domain T. But then we must loose associativity in operator/vector multiplication,

$$R(T|v\rangle) \equiv RT|v\rangle = 0 \qquad (RT)|v\rangle = |v\rangle\,.$$

These "associativity anomalies" [10] will continually recurr in our analysis and are closely related to midpoint ambiguities in the star algebra.

Let us now present some other choices of basis where the star product is simpler. Two sets of bases, sometimes called the "discrete even" and the "discrete odd" bases, have been

[4]The factor of $2n/(2m-1)$ for $R_{2m-1,2n}$ is taken following the convention of ref. [16] so that $\sum_{n=1}^{\infty} R_{2m-1,2n}T_{2n,2k-1} = \delta_{2m-1,2k-1}$. If we simply defined $R_{2m-1,2n} = \langle 2m-1|R|2n\rangle$ we would need to insert $2n/(2m-1)$ into the above sum to account for the nontrivial normalization of the basis $|2n\rangle, |2n-1\rangle$. Similar comments apply to the definition of v_{2n-1} in eq. (2.8).

[5]The fact that T is bounded but R is not will be clear when we present their spectrum [23].

[6]The normalization is chosen, as in [10] so that $\sum_{n=1}^{\infty} v_{2n-1}^2 = 1$.

JHEP03(2005)042

used extensively in the work of Bars and collaborators [16, 17]:

$$\begin{aligned}
&\text{even}: \quad && x_{2n} = x^D_{2n} && p_{2n-1} = \frac{1}{2}\sum_{m=1}^{\infty} p^D_{2m} T_{2m,2n-1} \\
&\text{odd}: \quad && x^D_{2n-1} = \frac{1}{2}\sum_{m=1}^{\infty} R_{2n-1,2m} x_{2m} && p^D_{2n-1} = p_{2m-1}\,.
\end{aligned} \tag{2.9}$$

Substituting these relations into eq. (2.2), bearing in mind eq. (2.6), we find[7]

$$[x^D_{2n}, p^D_{2m}]_\star = i\delta_{2n,2m} \qquad [x^D_{2n-1}, p^D_{2m-1}]_\star = i\delta_{2n-1,2m-1}\,. \tag{2.10}$$

An important assumption here is that the transformations eq. (2.9) are invertible. They are indeed invertible if T and R operate in the correct Hilbert spaces, but lacking a proper definition of the algebra of string fields its not clear whether this condition is likely to hold. For instance, it has been shown [17] that the perturbative vacuum is a well-defined functional in the discrete basis, but the sliver state [18] is not.

A strategy for dealing with these problems is to regulate/deform T into an invertible operator over its entire domain, while keeping the structure of the algebra eq. (2.10) fixed. This approach has been explored extensively by Bars et al. in references [10, 16, 17] and [19]–[21] and has been used in many concrete applications, particularly for calculating off-shell scattering amplitudes [22]. The main advantage of this formalism is that with an invertable T one can go much further towards formulating string field theory in the language of noncommutative field theory, in particular one can formulate the kinetic term purely in terms of a (regulated) star product. However, in deforming T the interactions are modified so that strings no longer split and join as smooth manifolds, and as a result it is difficult to demonstrate consistency and gauge invariance [17]. In this paper we will not regulate T and work directly in the exact theory.

For us, the most useful choice of basis on $\mathbb{P}$ will be the one where the noncommutativity parameter is diagonal, called the "continuous" or "kappa" basis [1, 23]. This basis is defined by introducing a continuous basis of vectors for $\mathcal{H}_{\rm odd}$ and $\mathcal{H}_{\rm even}$ respectively, labelled by an index κ,

$$|\kappa, \mathrm{o}\rangle \quad \kappa > 0 \qquad |\kappa, \mathrm{e}\rangle \quad \kappa > 0$$

which is delta function normalizable[8]:

$$\langle\kappa, \mathrm{e}|\kappa', \mathrm{e}\rangle = \delta(\kappa - \kappa') \;\; \kappa, \kappa' > 0 \qquad \langle\kappa, \mathrm{o}|\kappa', \mathrm{o}\rangle = \delta(\kappa - \kappa') \quad \kappa, \kappa' > 0\,.$$

Since $|\kappa, \mathrm{o}\rangle, |\kappa, \mathrm{e}\rangle$ are not normalizable they are not elements of $\mathcal{H}_{\rm odd}, \mathcal{H}_{\rm even}$ but can be used to span these Hilbert spaces in a distributional sense. In particular, in the κ basis $\mathcal{H}_{\rm even}$ and $\mathcal{H}_{\rm odd}$ are both the Hilbert space of square integrable functions on the interval $\kappa \in (0, \infty)$: $\langle f|f\rangle = \int_0^\infty d\kappa f^*(\kappa) f(\kappa)$. The κ basis is related to the mixed basis $|2n\rangle, |2n-1\rangle$

[7] Most of the work in the discrete Moyal formalism has focused on the even basis, probably for historical reasons [16]. The odd basis is just as good.

[8] These relations are modified at $\kappa = 0$ to $\langle 0, \mathrm{o}|\kappa, \mathrm{o}\rangle = \delta(\kappa)$ and $\langle 0, \mathrm{e}|\kappa, \mathrm{e}\rangle = 0$.

through the formulas,

$$\langle\kappa,\mathrm{e}|2n\rangle = \frac{\sqrt{2}}{\sqrt{2n}}v_{2n}(\kappa) \qquad \langle\kappa,\mathrm{o}|2n-1\rangle = -\sqrt{2}\sqrt{2n-1}v_{2n-1}(\kappa)\,, \tag{2.11}$$

where the functions $v_n(\kappa)$ ($n = 1,2,\dots$) are, up to a normalization, a set of orthogonal polynomials defined implicitly through the generating function [23, 32],

$$\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}}v_n(\kappa) = \frac{1-e^{-\kappa\tan^{-1}z}}{\kappa N(\kappa)}\,, \qquad N(\kappa) = \left(\frac{2}{\kappa}\sinh\frac{\pi\kappa}{4}\right)^{1/2}. \tag{2.12}$$

The v_ns have symmetry properties,

$$v_{2n-1}(\kappa) = v_{2n-1}(-\kappa) \qquad v_{2n}(\kappa) = -v_{2n}(-\kappa)$$

and are orthogonal:

$$\begin{aligned}
\sum_{m=1}^{\infty}v_m(k)v_m(k') &= \delta(k-k') \quad -\infty<\kappa,\kappa'<\infty \qquad \int_{-\infty}^{\infty}dk\; v_m(k)v_n(k) = \delta_{mn}\\
\sum_{m=1}^{\infty}v_{2m-1}(k)v_{2m-1}(k') &= \frac{1}{2}\delta(k-k') \quad 0<\kappa,\kappa'<\infty\\
\int_0^{\infty}dk\; v_{2m-1}(k)v_{2n-1}(k) &= \frac{1}{2}\delta_{2m-1,2n-1}\\
\sum_{m=1}^{\infty}v_{2m}(k)v_{2m}(k') &= \frac{1}{2}\delta(k-k') \quad 0<\kappa,\kappa'<\infty\\
\int_0^{\infty}dk\; v_{2m}(k)v_{2n}(k) &= \frac{1}{2}\delta_{2m,2n}\,.
\end{aligned} \tag{2.13}$$

A useful table of the properties of these functions can be found in the appendix. As was shown in ref. [1], the κ basis diagonalizes T in the sense that,

$$\langle\kappa,\mathrm{e}|T|\kappa',\mathrm{o}\rangle = \tanh\frac{\pi\kappa}{4}\delta(\kappa-\kappa')$$

and so the star product takes a simpler form. Specifically, defining the Moyal coordinates on $\mathbb{P}$,

$$\begin{aligned}
x(\kappa) &= \langle\kappa,\mathrm{e}|x\rangle = \sqrt{2}\sum_{n=1}^{\infty}v_{2n}(\kappa)\sqrt{2n}x_{2n}\\
y(\kappa) &= \langle\kappa,\mathrm{o}|p\rangle = -\sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}}p_{2n-1}\,.
\end{aligned} \tag{2.14}$$

The inner product on $\mathbb{P}$ becomes,

$$\langle u,u'\rangle = \int_0^{\infty}d\kappa[x(\kappa)x'(\kappa)+y(\kappa)y'(\kappa)]\,,$$

and the Moyal star commutation relations take the form:

$$[x(\kappa),y(\kappa')]_\star = 2i\theta(\kappa)\delta(\kappa-\kappa')\,, \tag{2.15}$$

where,

$$\theta(\kappa) = \tanh\frac{\pi\kappa}{4}$$

and the unwritten commutators vanish.

The main difference between the κ basis and discrete basis eq. (2.9) is that the in the κ basis the noncommutativity parameter has a zero at $\kappa = 0$:

$$\theta(\kappa)\delta(\kappa) = 0\,, \tag{2.16}$$

This zero corresponds precisely to the zero mode of T mentioned earlier:

$$\langle\kappa, \mathrm{o}|v\rangle = -\frac{4}{\sqrt{\pi}}\delta(\kappa)\,.$$

This means that the Moyal coordinates $x(\kappa), y(\kappa)$ become purely commutative towards $\kappa = 0$; $x(0)$ vanishes identically while $y(0)$ is proportional to the momentum of the left half of the string [1]:

$$P_L = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}p_{2n-1} = \int_0^{\pi/2} d\sigma p(\sigma) = -\frac{1}{\sqrt{\pi}}y(0)\,. \tag{2.17}$$

Since P_L is commutative it satisfies the property mentioned in the introduction,

$$P_L(\Psi * \Phi) = (P_L\Psi) * \Phi = \Psi * (P_L\Phi)\,. \tag{2.18}$$

By contrast, in the discrete basis the noncommutativity parameter is 1 by construction, so there should be no purely commutative coordinates like P_L in this description. Clearly this has something to do with the subtlety in inverting T, and the significance of this will become clearer in the next section.

Part of the confusion about P_L is that it is only commutative when multiplied with sufficiently well-behaved string fields [10]. Consider for example the commutator,

$$[\bar{x}, P_L]_\star\,,$$

where $\bar{x}$ is the position of the string center of mass relative to the midpoint:

$$\bar{x} = x_0 - x(\frac{\pi}{2}) = \sqrt{2}\sum_{n=1}^{\infty}(-1)^{n+1}x_{2n} = -2\int_0^{\infty}\frac{d\kappa}{\kappa N(\kappa)}x(\kappa)\,. \tag{2.19}$$

If we attempt evaluate the commutator in the mixed basis, we immediately encounter an ambiguous double sum,

$$[\bar{x}, P_L]_\star = i\frac{4}{\pi}\sum_{m,n=1}^{\infty}(-1)^{m+1}T_{2m,2n-1}\frac{(-1)^{n+1}}{2n-1}\,. \tag{2.20}$$

As noted in refs. [24, 10], the value of this sum depends on the order in which the summation is carried out. Explicitly, imposing a mode number cutoff N to P_L and N' to $\bar{x}$, one finds that:

$$\lim_{N'\to\infty}\lim_{N\to\infty}[\bar{x}(N'), P_L(N)]_\star = i\langle w|T|v\rangle = 0$$

$$\lim_{N\to\infty}\lim_{N'\to\infty}[\bar{x}(N'),P_L(N)]_\star = i\langle v|\bar{T}|w\rangle = i\,. \tag{2.21}$$

Here, in the notation of ref. [10], we introduced an even moded vector, $w_{2n} = \sqrt{2}(-1)^{n+1} = 2n\langle 2n|w\rangle$ satisfying

$$w_{2m}T_{2m,2n-1} = v_{2n-1}\,.$$

Thus, with a certain regularization, P_L may fail to be commutative.

It is useful to understand how this ambiguity manifests itself in the continuous basis, where the noncommutativity parameter is diagonal and there would not seem to be the possibility of an ambiguous "double sum." The key point is to notice that coordinates of the form

$$\int_0^\infty d\kappa\,[f(\kappa)y(\kappa)+g(\kappa)x(\kappa)]\,,$$

only have a well-defined Moyal-star commutator if $f(\kappa)$ and $g(\kappa)$ are sufficiently smooth functions. When calculating $[P_L,\bar{x}]_\star$, this is precisely what fails to happen — f and g in this case are distributions. We can see this by regulating the formulas for P_L and $\bar{x}$ as follows:

$$\begin{aligned}P_L(\omega) &= \frac{\sqrt{2}}{\pi i}\sum_{n=1}^\infty \frac{(i\tanh\omega)^{2n-1}}{2n-1}p_{2n-1} = -\frac{1}{\pi}\int_{-\infty}^\infty \frac{d\kappa}{N(\kappa)}\frac{\sin\omega\kappa}{\kappa}y(\kappa)\\ \bar{x}(\omega) &= -\sqrt{2}\sum_{n=1}^\infty (i\tanh\omega)^{2n}x_{2n} = -2\int_0^\infty \frac{d\kappa}{N(\kappa)}\frac{1-\cos\omega\kappa}{\kappa}x(\kappa)\,.\end{aligned} \tag{2.22}$$

Taking ω to infinity we should recover P_L and $\bar{x}$ as described before, but the coefficients in the integrand oscillate with infinite frequency and do not converge in the sense of functions (see figure 1). However, they *do* converge in the sense of distributions: $\sin\omega\kappa/\kappa$ converges to a delta function, $\delta(\kappa)$, while $(1-\cos\omega\kappa)/\kappa$ converges to the principal value, $\mathcal{P}\frac{1}{\kappa}$. Therefore, calculating $[\bar{x},P_L]_\star$ in the κ basis requires integrating a product of distributions — which is mathematically ill-defined. Still, we can calculate $[\bar{x}(\omega),P_L(\omega')]_\star$, for finite ω and ω', and take ω and ω' to infinity in different orders:

$$\begin{aligned}\lim_{\omega'\to\infty}\lim_{\omega\to\infty}[\bar{x}(\omega'),P_L(\omega)]_\star &= 2i\lim_{\omega'\to\infty}\frac{2}{\sqrt{\pi}}\int_0^\infty d\kappa\frac{\tanh\frac{\pi\kappa}{4}}{\kappa N(\kappa)}[1-\cos\omega'\kappa]\delta(\kappa) = 0\\ \lim_{\omega\to\infty}\lim_{\omega'\to\infty}[\bar{x}(\omega'),P_L(\omega)]_\star &= 2i\lim_{\omega\to\infty}\frac{2}{\pi}\int_{-\infty}^\infty d\kappa\frac{\tanh\frac{\pi\kappa}{4}}{2\kappa\sinh\frac{\pi\kappa}{2}}\sin\omega\kappa = i\,.\end{aligned}$$

Consistently, we find the same answers as we did imposing a mode cutoff in the mixed basis eq. (2.21).

3. Split strings in the continuous basis

Part of the goal of this paper is to understand how the continuity of the string modifies the naive "matrix product" picture of the star algebra at the string midpoint. Accordingly, in this section we will attempt to reformulate the star product directly as a matrix product via the split string formalism [24]–[26] and attempt to see how the description goes awry. Our discussion here parallels the discussion of ref. [10] but we will focus on the split string

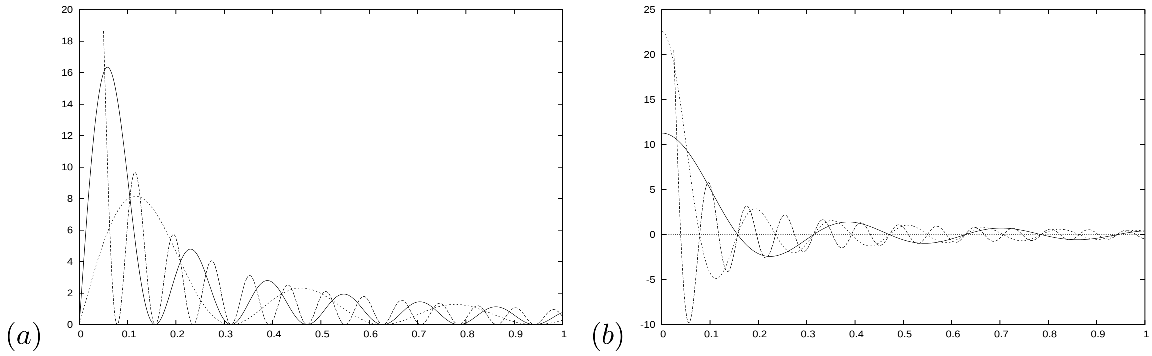

Figure 1: Sequence of increasingly oscillatory functions, eq. (2.22), converging to (a) the principal value distribution, (b) the delta function, graphed in this case for $\omega = 20, 40, 80$.

description of the κ-basis. Unlike the discrete basis, the continuous basis was originally derived from the three string vertex $\langle V_3|$ and its "position space" interpretation as a half-string product has not been discussed[9] and is of interest in its own right.

Often it is convenient to represent multiplication in the algebra of functions on non-commutative $\Re^2$ as a nonlocal Moyal product between otherwise commutative functions on $\Re^2$. However, another useful perspective is the "dipole" matrix-like representation of the algebra. This can be related to the Moyal perspective as follows: Given a function $\psi(x, p)$ with $[x, p]_\star = 2i\theta$, define,

$$\hat{\psi}(\hat{l}, \hat{r}) = \bar{\psi}(\bar{l}, \bar{r}) = \int_{-\infty}^{\infty} dp e^{ipz} \psi(x, p)\,,$$

where

$$\begin{aligned} \hat{l} &= \theta z + x \qquad & \hat{r} &= -\theta z + x \\ \bar{l} &= z + \frac{x}{\theta} \qquad & \hat{r} &= -z + \frac{x}{\theta}\,. \end{aligned} \tag{3.1}$$

A simple excercise [16] shows that calculating the "matrix product" of these objects,

$$\begin{aligned} \hat{A} \star \hat{B}(\hat{l}, \hat{r}) &= \theta \int_{-\infty}^{\infty} dw \hat{A}(\hat{l}, w) \hat{B}(w, \hat{r}) \\ \bar{A} \star \bar{B}(\bar{l}, \bar{r}) &= \int_{-\infty}^{\infty} dw \bar{A}(\bar{l}, w) \bar{B}(w, \bar{r})\,, \end{aligned} \tag{3.2}$$

is isomorphic to calculating the Moyal product of $A(x, p)$ and $B(x, p)$. Note that we have distinguished two types of matrix product when $\theta \neq 1$, by writing fields/coordinates alternatively with a hat or a bar. The significance of this distinction will be clear in a moment.

[9] Recently, the work of ref. [27] explores some aspects of the split string formalism from the perspective of the κ-basis.

We can easily follow this recipe to construct a split string representation of the kappa basis. Write,

$$
\begin{aligned}
\Psi[x_n] &= \int \prod_{n=1}^{\infty} \left(\frac{dp_{2n-1}}{2\pi} e^{ip_{2n-1}x_{2n-1}} \right) \Psi[x_{2n}, p_{2n-1}] \\
&= \int [dy(\kappa)]_1 \exp \left(\int_0^\infty d\kappa y(\kappa) z(\kappa) \right) \Psi^M[x(\kappa), y(\kappa)] \\
&= \hat{\Psi}[\hat{l}(\kappa), \hat{r}(\kappa)] = \bar{\Psi}[\bar{l}(\kappa), \bar{r}(\kappa)] \,,
\end{aligned}
$$

with

$$
\begin{aligned}
\hat{l}(\kappa) &= \theta(\kappa) z(\kappa) + x(\kappa)\,, \qquad \hat{r}(\kappa) = -\theta(\kappa) z(\kappa) + x(\kappa) \\
\bar{l}(\kappa) &= z(\kappa) + \frac{x(\kappa)}{\theta(\kappa)} \qquad \bar{r}(\kappa) = -z(\kappa) + \frac{x(\kappa)}{\theta(\kappa)}\,.
\end{aligned}
\tag{3.3}
$$

We can then calculate the open string star product as,

$$
\begin{aligned}
\hat{A} * \hat{B}[\hat{l}(\kappa), \hat{r}(\kappa)] &= \mathcal{N} \int [dw(\kappa)]_2 \hat{A}[\hat{l}(\kappa), w(\kappa)] \hat{B}[w(\kappa), \hat{r}(\kappa)] \\
\bar{A} * \bar{B}[\bar{l}(\kappa), \bar{r}(\kappa)] &= \mathcal{N} \int [dw(\kappa)]_3 \bar{A}[\bar{l}(\kappa), w(\kappa)] \bar{B}[w(\kappa), \bar{r}(\kappa)]\,.
\end{aligned}
\tag{3.4}
$$

The measures in the previous formulae are normalized so that,

$$
\begin{aligned}
1 &= \int [dw(\kappa)]_1 \exp \left[-\frac{1}{2\pi} \int_0^\infty d\kappa d\kappa' K_{-1}(\kappa, \kappa') w(\kappa) w(\kappa') \right] \\
&= \int [dw(\kappa)]_2 \exp \left[-\frac{\pi}{2} \int_0^\infty d\kappa d\kappa' K_{-1}(\kappa, \kappa') w(\kappa) w(\kappa') \right] \\
&= \int [dw(\kappa)]_3 \exp \left[-\frac{\pi}{2} \int_0^\infty d\kappa d\kappa' \frac{K_{-1}(\kappa, \kappa')}{\theta(\kappa)\theta(\kappa')} w(\kappa) w(\kappa') \right].
\end{aligned}
\tag{3.5}
$$

These expressions can all be derived by translating from the corresponding measures in the mode basis. The kernel $K_{-1}(k, k')$ is given by the sum $\sum_{n=1}^{\infty}(2n-1)^{-1} v_{2n-1}(\kappa) v_{2n-1}(\kappa')$. An explicit formula for K_{-1} was found in [28, 29], but we will not need it.

It is useful to relate eq. (3.3) to other half-string bases. A simple computation shows,

$$
\begin{aligned}
\hat{l}(\kappa) &= \sqrt{2} \sum_{n=1}^{\infty} \sqrt{2n} v_{2n}(\kappa) l_{2n} = -\tanh \frac{\pi\kappa}{4} \sqrt{2} \sum_{n=1}^{\infty} \sqrt{2n-1} v_{2n-1}(\kappa) l_{2n-1}\,, \\
\bar{l}(\kappa) &= \coth \frac{\pi\kappa}{4} \sqrt{2} \sum_{n=1}^{\infty} \sqrt{2n} v_{2n}(\kappa) l_{2n} = -\sqrt{2} \sum_{n=1}^{\infty} \sqrt{2n-1} v_{2n-1}(\kappa) l_{2n-1}\,,
\end{aligned}
\tag{3.6}
$$

where l_{2n} denotes even cosine half-string Fourier modes and l_{2n-1} denotes odd cosine half-string Fourier modes:

$$
l_{2n} = \frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} d\sigma x(\sigma) \cos 2n\sigma \qquad l_{2n-1} = \frac{2\sqrt{2}}{\pi} \int_0^{\pi/2} d\sigma x(\sigma) \cos(2n-1)\sigma\,.
$$

We can further relate $\hat{l}(\kappa), \bar{l}(\kappa)$ to the left half of the string, $l(\sigma) = x(\sigma),\ \sigma \in [0, \frac{\pi}{2})$:

$$\begin{aligned}\hat{l}(\kappa) &= -\frac{2}{\sqrt{\pi}} \frac{\sinh \frac{\pi\kappa}{4}}{N(\kappa)} \int_0^{\pi/2} d\sigma \sec\sigma M(\kappa,\sigma) l(\sigma) \\ \bar{l}(\kappa) &= -\frac{2}{\sqrt{\pi}} \frac{\cosh \frac{\pi\kappa}{4}}{N(\kappa)} \int_0^{\pi/2} d\sigma \sec\sigma M(\kappa,\sigma) l(\sigma)\,,\end{aligned} \tag{3.7}$$

where $M(\kappa,\sigma)$ is derived by using the generating function eq. (2.12) to evaluate the sum,

$$\begin{aligned}M(\kappa,\sigma) &= \frac{2N(\kappa)\cos\sigma}{\sqrt{\pi}\cosh\frac{\pi\kappa}{4}} \sum_{n=1}^{\infty} \sqrt{2n-1}\, v_{2n-1}(\kappa) \cos(2n-1)\sigma \\ &= \frac{1}{\sqrt{\pi}} \cos\left(\frac{\kappa}{2} \tanh^{-1} \sin\sigma\right).\end{aligned} \tag{3.8}$$

The M's obey orthogonality relations:

$$\begin{aligned}\int_0^{\pi/2} d\sigma \sec\sigma M(\kappa,\sigma) M(\kappa',\sigma) &= \delta(\kappa-\kappa') \\ \int_0^{\infty} d\kappa M(\kappa,\sigma) M(\kappa,\sigma') &= \cos\sigma\,\delta(\sigma-\sigma')\,.\end{aligned} \tag{3.9}$$

At first sight, the functions $M(\kappa,\sigma)$ seem exotic. It seems reasonable to expand the half string into a basis of cosines, as is usually done; why expand a half string in a basis of $M(\kappa,\sigma)$s? Actually, the M's are special: they are eigenfunctions of the midpoint preserving reparameterization generator,

$$\mathcal{K}_1 = L_1 + L_{-1} = -2i\cos\sigma \frac{\partial}{\partial\sigma}\,.$$

Specifically,

$$(\mathcal{K}_1)^2 M(\kappa,\sigma) = -\kappa^2 M(\kappa,\sigma)\,.$$

This is not surprising, since the kappa basis is defined by the eigenvectors of the Neumann coefficients, which in turn were derived in ref. [23] from the observation that a matrix K_1, describing the action of $\mathcal{K}_1$ on $\langle V_3|$, possessed the same eigenvectors as the Neumann coefficients.

$\mathcal{K}_1$ acts like a generator of translations, but translations whose length becomes increasingly shrunk towards the midpoint; so in a sense the midpoint appears infinitely "far away." Correspondingly, the eigenfunctions $M(\kappa,\sigma)$ look locally like sine waves, but their frequency is position-dependent and blows up towards $\sigma = \frac{\pi}{2}$. (See figure 2). The situation is therefore quite different from the discrete even/odd Moyal bases, which impose Neumann/Dirichlet boundary conditions at the midpoint, respectively; the value of M and its derivative at the midpoint is completely undefined. In the continuous basis we should think of the midpoint boundary condition in the sense of a singular Sturm-Liouville system; one should require that the eigenfunctions remain bounded as σ goes to $\frac{\pi}{2}$. This effectively fixes κ to be real.

JHEP03(2005)042

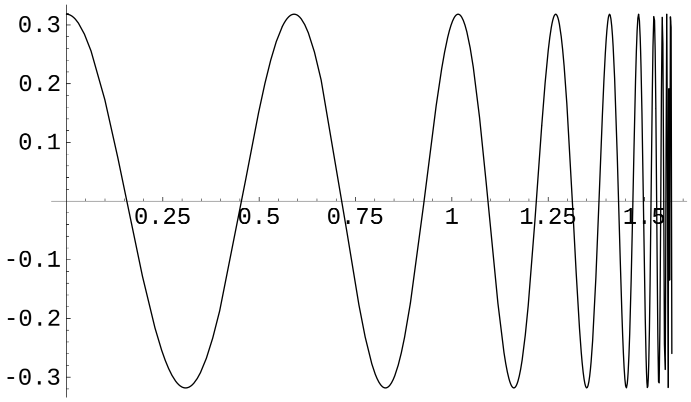

Figure 2: An example, $M(10, \sigma)$.

The fundamental question in the split string formalism is whether one's choice of half string variables can represent the configuration $l(\sigma) = 1$. The relevance of this question should be explained. First, note that since we are working at zero momentum, we can for definiteness fix the string midpoint to be at zero: $x(\frac{\pi}{2}) = 0$. A string configuration with a kink discontinuity at the midpoint is,

$$x(\sigma) = \begin{cases} +1 & \text{for } \sigma \in \left[0, \frac{\pi}{2}\right) \\ 0 & \text{for } \sigma = \frac{\pi}{2} \\ -1 & \text{for } \sigma \in \left(\frac{\pi}{2}, \pi\right] \end{cases} .$$

To describe this we need a basis capable of describing the function $l(\sigma) = 1$. There is reason to believe that this configuration may be useful in string field theory, even though it looks somewhat singular; for instance it was shown in ref. [11] that the sliver functional, which is the canonical projector solution of VSFT, is independent of transformations which break a string at its midpoint. These considerations were further studied in ref. [23, 30, 31]. However, if we choose a half-string basis which *can* describe $l(\sigma) = 1$ we encounter another problem: the basis accommodates configurations like:

$$x(\sigma) = \begin{cases} +1 & \text{for } \sigma \in \left[0, \frac{\pi}{2}\right) \\ 0 & \text{for } \sigma = \frac{\pi}{2} \\ +1 & \text{for } \sigma \in \left(\frac{\pi}{2}, \pi\right] \end{cases} . \tag{3.10}$$

This unphysical situation describes a string which has become disconnected from its midpoint. There is no evidence that such configurations have any realization in string field theory, in fact there is some evidence to the contrary [23]. But now we are stuck in a catch-22: if we cannot describe $l(\sigma) = 1$ the string has too few degrees of freedom, but if we can it has too many.[10] Thus the split string formalism cannot give a completely accurate representation of the string's degrees of freedom or of the star product, though for certain applications these subtleties may not matter.

This problem is clearly independent of one's choice of basis, and in particular can be seen in the κ basis. Plugging $l(\sigma) = 1$ into eq. (3.7),

$$\begin{aligned}
\bar{l}(\kappa) &= -\frac{2}{\sqrt{\pi}} \frac{\cosh \frac{\pi\kappa}{4}}{N(\kappa)} \int_0^{\pi/2} d\sigma \sec\sigma M(\kappa, \sigma) = -\frac{2}{\pi}\delta(\kappa) \\
\hat{l}(\kappa) &= -\frac{2}{\sqrt{\pi}} \frac{\sinh \frac{\pi\kappa}{4}}{N(\kappa)} \int_0^{\pi/2} d\sigma \sec\sigma M(\kappa, \sigma) = -\frac{2}{\pi} \sinh \frac{\pi\kappa}{4}\delta(\kappa) = 0\,.
\end{aligned}$$

Apparently, $\bar{l}(\kappa)$ can represent $l(\sigma) = 1$ but $\hat{l}(\kappa)$ can't. Neither situation is satisfactory. However, it is interesting to compare the above result to what happens in the discrete even/odd basis, which expands the half string in even/odd cosine modes l_{2n}, l_{2n-1}. Even cosine modes cannot represent $l(\sigma) = 1$ since they impose Neumann boundary conditions at the midpoint, whereas odd cosine modes, imposing Dirichlet boundary conditions, can [10]. By contrast, in the continuous basis the distinction comes from normalisation of the continuum of half string modes, rather than from a boundary condition at the midpoint.

It is important to emphasize the split string formalism is *not* equivalent to the continuous Moyal product described by eq. (2.15); the half-string degrees of freedom $\bar{l}(\kappa), \bar{r}(\kappa)$ and $\hat{l}(\kappa), \hat{r}(\kappa)$ are not related in an invertable way to $x(k), y(\kappa)$. This is actually not surprising, since $\theta(\kappa)$ has a continuous spectrum around zero and a matrix product is only isomorphic to a Moyal product when $\theta \neq 0$; for $\theta = 0$ the Moyal product is just a commutative product of functions, not a matrix product. Therefore to make the translation to the split string formalism we implicitly had to make a change of basis which normalized the noncommutativity parameter to 1. There are essentially two distinct ways to do this, corresponding to our two half string bases. We can define $\hat{y}(\kappa) = y(\kappa)/\theta(\kappa)$ so that

$$[\hat{y}(\kappa), x(\kappa)]_\star = 2i\delta(\kappa - \kappa')\,,$$

but then the position coordinate congugate to $\hat{y}(\kappa)$,

$$\hat{z}(\kappa) = \theta(\kappa) z(\kappa)$$

vanishes identically at $\kappa = 0$; in particular the configuration $z(\kappa) = \delta(\kappa)$, corresponding to a string with a kink at the midpoint, is lost in this basis. Alternatively we can define $\bar{x}(\kappa) = x(\kappa)/\theta(\kappa)$ so that

$$[y(\kappa), \bar{x}(\kappa)]_\star = 2i\delta(\kappa - \kappa')\,,$$

[10] Another interpretation of eq. (3.10) advocated in ref. [24] is that it really represents a string which has been translated to $x(\sigma) = 1$, and $x(\pi/2) = 0$ is a residual artifact. We belive that this interpretation is problematic, both because the string midpoint is represented in a singular fashion and because similar problems arise when the string endpoints are fixed with Dirichlet boundary conditions to a D-brane, so there does not seem to be anything (like nonzero momentum) which could reinterpret the unphysical degree of freedom.

but in this basis we gain a new nonvanishing coordinate at $\kappa = 0$,

$$\bar{x}(0) = \lim_{\kappa\to 0} \frac{x(\kappa)}{\tanh\frac{\pi\kappa}{4}} = \frac{4}{\pi}x'(0)\,.$$

It turns out that this new coordinate is soley responsible for describing the isolated midpoint configuration:

$$\bar{x}(\kappa) = \frac{1}{2}[\bar{l}(\kappa) + \bar{r}(\kappa)] = -\frac{2}{\pi}\delta(\kappa)\,.$$

Modulo some Fourier transforms and a change of basis, all of this is essentially a reexpression of the same problem we encountered in section 2 in defining the discrete basis: T has a zero mode so we cannot invert it to normalize the noncommutativity parameter to one.

In connection to the unphysical coordinate $\bar{x}(0)$ we mention a related issue with defining the spectrum of the Neumann coefficints. In ref. [23] it was shown that the Neumann matrix $M_{mn} = (-1)^m V^{11}_{mn}$ has a doubly degenerate spectrum of eigenvectors, $v_{2n}(\kappa)$ and $v_{2n-1}(\kappa)$, for κ strictly greater than zero; for $\kappa = 0$, however, M only has *one* twist odd eigenvector, with eigenvalue $-\frac{1}{3}$:

$$v_{2n-1}(0) = \frac{1}{\sqrt{\pi}}\frac{(-1)^{n+1}}{\sqrt{2n-1}}\,.$$

since the twist even eigenvector $v_{2n}(0)$ vanishes identically. However, one might try to recover a twist even eigenvector at $\kappa = 0$ by normalizing the eigenvectors by $1/\kappa$. At $\kappa = 0$ we then formally derive a new eigenvector,

$$v'_{2n}(0) = \frac{1}{\sqrt{\pi}}\frac{(-1)^{n+1}}{\sqrt{2n}}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}\right), \tag{3.11}$$

but ref. [23] gave several reasons for excluding this vector from the spectrum of M. Our discussion brings a little more insight into this question, since the formal eigenvector $v'_{2n}(0)$ would be responsible for describing the isolated midpoint degree of freedom:

$$\bar{x}(0) = \frac{4}{\pi}x'(0) = \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\sqrt{2n}v'_{2n}(0)x_{2n}\,.$$

We do not expect this configuration to play a role in the theory, so it seems reasonable to exclude $v'_{2n}(0)$ from the spectrum of M. Accordingly, all "well-behaved" string fields should be independent of $x'(0)$:

$$\frac{\delta\Psi}{\delta x'(0)} = \lim_{\kappa\to 0}\kappa\frac{\delta\Psi}{\delta x(\kappa)} = 0\,. \tag{3.12}$$

Since $\delta/\delta x'(0)$ is a derivation, this constraint is preserved by star multiplication and so is consistent with the structure of the theory. It is worth noting that the coordinate $\bar{x}$ from eq. (2.19) fails to satisfy this criterion,

$$\frac{\delta\bar{x}}{\delta x'(0)} = \lim_{\kappa\to 0}\kappa\frac{\delta\bar{x}}{\delta x(\kappa)} = -\frac{2}{\sqrt{\pi}}$$

which is basically why the commutator $[\bar{x}, P_L]_\star$ can be nonzero. This makes sense, since $\delta/\delta x'(0)$ translates the midpoint away from the rest of the string, and this obviously effects the distance between the midpoint and the center of mass.

JHEP03(2005)042

4. Ghosts

So far our analysis has dealt with the matter sector exclusively. In bosonized form [24, 15], the star algebra of ghosts is identical[11] to the matter sector, eq. (2.2). However, in the bc ghost language, the algebra looks quite different [3, 12, 22], leading to some important algebraic structures beyond what we've seen. Define the anticommuting ghost fields,

$$
\begin{aligned}
b(\sigma) &= i\sqrt{2}\sum_{n=1}^{\infty} x_n \sin n\sigma \qquad x_n = \frac{i}{\sqrt{2}}(b_n - b_{-n}) \\
\pi_c(\sigma) &= \frac{1}{\pi}\left(b_0 + \sqrt{2}\sum_{n=1}^{\infty} q_n \cos n\sigma\right) \qquad q_n = \frac{1}{\sqrt{2}}(b_n + b_{-n}) \\
c(\sigma) &= c_0 + \sqrt{2}\sum_{n=1}^{\infty} y_n \cos n\sigma \qquad y_n = \frac{1}{\sqrt{2}}(c_n + c_{-n}) \\
\pi_b(\sigma) &= -\frac{i\sqrt{2}}{\pi}\sum_{n=1}^{\infty} p_n \sin n\sigma \qquad p_n = \frac{i}{\sqrt{2}}(c_n - c_{-n}) .
\end{aligned}
$$

b_n and c_n are the mode oscillators of the bc ghost conformal field theory satisfying $\{b_n, c_{-m}\} = \delta_{m,n}$ (note that our use of the x_n and p_n differs from previous sections). The discussion is simplified by restricting to Siegel gauge [12], where we will choose a representation of the the string field as a functional of the eigenvalues of $u = \{x_{2n-1}, p_{2n}, y_{2n-1}, q_{2n}\}$ for $n \geq 1$. Just as in the matter sector, these eigenvalues are a set of coordinates a linear space $\mathbb{P}_{gh}$ with a real symmetric inner product,

$$\langle u, u'\rangle = i\sum_{n=1}^{\infty}\left(x_{2n-1}y'_{2n-1} - y_{2n-1}x'_{2n-1} - q_{2n}p'_{2n} + p_{2n}q'_{2n}\right) .$$

We can then calculate the reduced star product [13] (= b_0 times the full star product) as:

$$\Psi *_{b_0} \Phi[u] = \mathcal{N}_{gh}\Psi \star \Phi[u] , \tag{4.1}$$

where $\mathcal{N}_{gh}$ is a normalization [12] and $\star$ is a fermionic, non-anticommutative, "Moyal product" satisfying,

$$\{y_{2m-1}, q_{2n}\}_\star = 2R_{2m-1,2n} \qquad \{x_{2m-1}, p_{2n}\}_\star = 2\frac{2m-1}{2n}R_{2m-1,2n} , \tag{4.2}$$

where $\{,\}_\star$ is the Moyal anticommutator and R from eq. (2.6) plays the role of the non-anticommutativity parameter.

As in the matter sector, the above "mixed" basis can be thought of as a choice of basis on $\mathcal{H}_{\rm odd}$, $\mathcal{H}_{\rm even}$ corresponding to,

$$
\begin{aligned}
x_{2m-1} &= -\langle 2m-1|x\rangle \qquad y_{2m-1} = -\frac{1}{2m-1}\langle 2m-1|y\rangle \\
p_{2n} &= \langle 2n|p\rangle \qquad q_{2n} = -2n\langle 2n|q\rangle
\end{aligned} \tag{4.3}
$$

[11] Restricting to the subalgebra of fields with ghost momentum $-3/2$ is equivalent to studying zero momentum in the matter sector

so that

$$\langle u, u' \rangle = i \left[\langle x|y' \rangle - \langle y|x' \rangle + \langle q|p' \rangle - \langle p|q' \rangle \right].$$

However, the star product is simplified by making a transformation to the kappa basis:

$$\begin{aligned} x_{\mathrm{e}}(\kappa) &= \langle \kappa, \mathrm{e}|q \rangle = -\sqrt{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}} v_{2n}(\kappa) q_{2n} \\ x_{\mathrm{o}}(\kappa) &= \langle \kappa, \mathrm{o}|x \rangle = \sqrt{2} \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n-1}} v_{2n-1}(\kappa) x_{2n-1} \\ y_{\mathrm{e}}(\kappa) &= \langle \kappa, \mathrm{e}|p \rangle = \sqrt{2} \sum_{n=1}^{\infty} \sqrt{2n} v_{2n}(\kappa) p_{2n} \\ y_{\mathrm{o}}(\kappa) &= \langle \kappa, \mathrm{o}|y \rangle = \sqrt{2} \sum_{n=1}^{\infty} \sqrt{2n-1} v_{2n-1}(\kappa) y_{2n-1}. \end{aligned} \tag{4.4}$$

In this basis, the inner product on $\mathbb{P}_{gh}$ takes the form,

$$\langle u, u' \rangle = i \int_0^{\infty} d\kappa \left[\vec{x}(\kappa) \cdot \sigma_z \cdot \vec{y}(\kappa)' - \vec{y}(\kappa) \cdot \sigma_z \cdot \vec{x}(\kappa)' \right],$$

where we defined $\vec{x} = (x_{\mathrm{e}}, x_{\mathrm{o}}), \vec{y} = (y_{\mathrm{e}}, y_{\mathrm{o}})$ and σ_z is the zth Pauli matrix. The algebra eq. (4.2) now takes the form,

$$\{\vec{x}(\kappa), \vec{y}(\kappa')\}_\star = 2i\sigma_y g(\kappa)\ \delta(\kappa - \kappa'), \tag{4.5}$$

where

$$g(\kappa) = \coth \frac{\pi\kappa}{4}.$$

Note, in particular, that $g(\kappa)$ has a pole at $\kappa = 0$, and no zeros.

As in the matter sector, all of the interesting structure in this algebra lies at $\kappa = 0$. However, the physical significance of this structure seems obscure. What does it mean that the $\kappa = 0$ coordinates have infinite non-anticommutativity? The odd Moyal coordinates at $\kappa = 0$,

$$x_{\mathrm{o}}(0) = \sqrt{\frac{2}{\pi}} \left(\hat{x}_1 - \frac{1}{3}\hat{x}_3 + \frac{1}{5}\hat{x}_5 - \cdots \right) \qquad y_{\mathrm{o}}(0) = \sqrt{\frac{2}{\pi}} \left(\hat{y}_1 - \hat{y}_3 + \hat{y}_5 - \cdots \right),$$

do not correspond to any particularly simple string degree of freedom. Even more puzzling, the even coordinates *vanish* at $\kappa = 0$, yet somehow manage to have infinite non-anticommutativity with the odd coordinates! Surely $0 * x_{\mathrm{o}}(0) = x_{\mathrm{o}}(0) * 0 = 0$?

Part of the confusion here amounts to a simple problem of domains. The anticommutator eq. (4.4) must be thought of as an equation relating distributions on the appropriate space of test functions. In this case, the space should be square integrable functions on the interval $\kappa \in [0, \infty)$ which vanish linearly[12] (or faster) at $\kappa = 0$: $\lim_{\kappa \to 0} |f(\kappa)/\kappa| < \infty$. On this space, define a pair of distributions,

$$g \cdot \delta(\kappa, \kappa') = g \cdot \bar{\delta}_\kappa(\kappa') = g \cdot \delta_{\kappa'}(\kappa)$$

[12] This space can be thought of as the domain of the linear operator R, which is the image of $\mathcal{H}_{\mathrm{odd}}$ under the action of T.

so that,

$$\langle g\cdot\delta_\kappa,\ f\rangle = \int_0^\infty d\kappa' g\cdot\delta(\kappa',\kappa)f(\kappa') = \begin{cases} g(\kappa)f(\kappa) & \text{for } \kappa>0 \\ \dfrac{2}{\pi}f'(0) & \text{for } \kappa=0 \end{cases}$$

$$\langle g\cdot\bar\delta_\kappa,\ f\rangle = \int_0^\infty d\kappa' g\cdot\delta(\kappa,\kappa')f(\kappa') = \begin{cases} g(\kappa)f(\kappa) & \text{for } \kappa>0 \\ 0 & \text{for } \kappa=0 \end{cases}. \tag{4.6}$$

Both of these distributions are *almost* equivalent to $g(\kappa)\delta(\kappa-\kappa')$, but crucially differ at $\kappa,\kappa'=0$; apparently, $g\cdot\delta(\kappa,\kappa')$ is not symmetric when considered as a distribution in κ versus κ'. With these definitions, we should more precisely write the star algebra in the ghost sector as,

$$\{x_{\rm e}(\kappa), y_{\rm o}(\kappa')\}_\star = -\{y_{\rm e}(\kappa), x_{\rm o}(\kappa')\}_\star = 2g\cdot\delta(\kappa,\kappa') \tag{4.7}$$

with unwritten anti-commutators vanishing. The nontrivial component here is the value of the these expressions at $\kappa,\kappa'=0$. Setting $\kappa=0$ for the even coordinates, we see now that both sides of this equation vanish unambiguously as a result of the definition eq. (4.6). To see that eq. (4.6) gives the right result for the odd coordinates at $\kappa'=0$, let us use the mixed basis to calculate the anticommutator,[13]

$$\{y_{\rm o}(0), q_{2n}\}_\star = -2\sqrt{\frac{2}{\pi}}\sum_{m=1}^\infty(-1)^m R_{2m-1,2n} = -\frac{8\sqrt2}{\pi}\sqrt{2n}\ v'_{2n}(0)\,.$$

On the other hand, using eq. (4.6) we have,

$$\begin{aligned}\{y_{\rm o}(0), q_{2n}\}_\star &= -\sqrt2\sqrt{2n}\int_0^\infty d\kappa v_{2n}(\kappa)\{y_{\rm o}(0), x_{\rm e}(k)\}_\star \\ &= -\sqrt2\sqrt{2n}\int_0^\infty d\kappa v_{2n}(\kappa)g\cdot\delta(\kappa,0) \\ &= -\frac{8\sqrt2}{\pi}\sqrt{2n}\ v'_{2n}(0)\,.\end{aligned}$$

So the definition works out consistently. As an aside we mention that, in the matter sector, the correct space of test functions for the commutator eq. (2.15) is bounded C^∞ functions

[13]This result may be derived from the following manipulation:

$$\begin{aligned}\{y_{\rm o}(0), \sum_{n=1}^\infty z^{2n}q_{2n}\}_\star &= -2\sqrt{\frac{2}{\pi}}\sum_{m,n=1}^\infty(-1)^m z^{2n}R_{2m-1,2n} \\ &= \frac{4}{\pi}\sqrt{\frac{2}{\pi}}\int_0^{\pi/2}d\sigma(\sec\sigma)\left(\frac{2z^2}{(1+z^2)(1-z^2)}\frac{\cos^2\sigma}{1+(\frac{2z}{1-z^2}\sin\sigma)^2}\right) \\ &= \frac{4}{\pi}\sqrt{\frac{2}{\pi}}\frac{z}{1+z^2}\int_0^{\frac{2z}{1-z^2}}\frac{du}{1+u^2} \\ &= \frac{8}{\pi}\sqrt{\frac{2}{\pi}}\frac{z\tan^{-1}z}{1+z^2}\,.\end{aligned}$$

Taylor expanding in z and recalling eq. (3.11) we get the answer.

on $\kappa \in [0,\infty)$. The zero in $\tanh\frac{\pi\kappa}{4}$ might allow one to extend to a space of test functions with a pole at the boundary, but as we've seen this has the effect of introducing the spurious coordinate $x'(0)$ into the theory.

Fixing domains brings a little more rigor into our formulae, but still the physical interpretation of the pole and the corresponding coordinates at $\kappa = 0$ needs clarification. Writing $x_{\rm o}(\kappa)$ and $y_{\rm o}(\kappa)$ in terms of $c(\sigma)$ and $b(\sigma)$, we find

$$\begin{aligned} y_{\rm o}(\kappa) &= \frac{\cosh\frac{\pi\kappa}{4}}{\sqrt{\pi}N(\kappa)}\int_0^\pi d\sigma \sec\sigma M(\kappa,\sigma)\; c(\sigma) \\ x_{\rm o}(\kappa) &= \frac{\cosh\frac{\pi\kappa}{4}}{\sqrt{\pi}N(\kappa)}\int_0^\pi d\sigma \sec\sigma M(\kappa,\sigma)\; \tilde{b}(\sigma)\,, \end{aligned}$$

where,

$$\tilde{b}(\sigma) = \int_{\pi/2}^{\sigma} d\sigma' b(\sigma')\,.$$

With this formula it is easy to check that $x_{\rm o}(0)$ and $y_{\rm o}(0)$ are necessary to describe string configurations where $c(\sigma)$ and $\tilde{b}(\sigma)$ have a midpoint discontinuity. A midpoint discontinuity in $\tilde{b}(\sigma)$ corresponds to a delta function in $b(\sigma)$ at the midpoint: $b(\sigma) = \delta(\sigma - \frac{\pi}{2})$. Specifically, suppose that $c(\sigma)$ is a step function which jumps from 1 to -1 at the midpoint. Then,

$$y_{\rm o}(\kappa) = \frac{2\cosh\frac{\pi\kappa}{4}}{\sqrt{\pi}N(\kappa)}\int_0^{\pi/2} d\sigma \sec\sigma M(\kappa,\sigma) = \frac{2}{\sqrt{\pi}}\delta(\kappa)\,,$$

so the configuration is described solely with $y_{\rm o}(0)$. From this it follows that,

$$\begin{aligned} \frac{\delta}{\delta y_{\rm o}(0)} &= \frac{\sqrt{\pi}}{2}\left(\int_0^{\pi/2} d\sigma \pi_c(\sigma) - \int_{\pi/2}^{\pi} d\sigma \pi_c(\sigma)\right) \equiv \sqrt{\pi}\Pi_{c,L} \\ \frac{\delta}{\delta x_{\rm o}(0)} &= \frac{\sqrt{\pi}}{2}\pi_b(\frac{\pi}{2})\,. \end{aligned} \tag{4.8}$$

Varying with respect to $x_{\rm o}(0)$ gives the half-string momentum of the c ghost, while varying with respect to $y_{\rm o}(0)$ gives the b ghost momentum at the midpoint.

Before discussing the meaning of the pole in $g(\kappa)$, it is important to emphasize that we should not redefine our basis so that the ghost nonanticommutativity parameter is normalized to one. Since $g(\kappa)$ has no zeros one might think that there is no obstruction to doing this, but it turns out that the pole creates the same problems for the ghost sector as the zero in $\theta(\kappa)$ does for the matter sector. There are two ways we could try to normalize the nonanticommuativity parameter to one. We can define $\hat{y}_{\rm o}(\kappa) = y_{\rm o}(\kappa)/g(\kappa)$ and $\hat{x}_{\rm o}(\kappa) = x_{\rm o}(\kappa)/g(\kappa)$ so that

$$\{x_{\rm e}(\kappa), \hat{y}_{\rm o}(\kappa')\}_\star = -\{y_{\rm e}(\kappa), \hat{x}_{\rm o}(\kappa')\} = 2\delta(\kappa-\kappa')\,.$$

However, due to the pole in $g(\kappa)$, $\hat{y}_{\rm o}(\kappa), \hat{x}_{\rm o}(\kappa)$ vanish identically at $\kappa = 0$, and we loose the ability to describe the string when $c(\sigma)$ has a kink and $b(\sigma)$ has a delta function at the midpoint. On the other hand we could define $\bar{y}_{\rm e}(\kappa) = y_{\rm e}(\kappa)/g(\kappa)$ and $\bar{x}_{\rm e}(\kappa) = x_{\rm e}(\kappa)/g(\kappa)$ so that

$$\{\bar{x}_{\rm e}(\kappa), y_{\rm o}(\kappa')\}_\star = -\{\bar{y}_{\rm e}(\kappa), x_{\rm o}(\kappa')\} = 2\delta(\kappa-\kappa')\,.$$

JHEP03(2005)042

The position coordinates conjugate to the momenta $\bar{x}_{\rm e}(\kappa), \bar{y}_{\rm e}(\kappa)$ are,

$$\bar{q}_{\rm e}(\kappa) = g(\kappa) q_{\rm e}(\kappa) \qquad \bar{p}_{\rm e}(\kappa) = g(\kappa) p_{\rm e}(\kappa) \tag{4.9}$$

and

$$p_{\rm e}(\kappa) = -\sqrt{2}\sum_{n=1}^{\infty}\sqrt{2n}\, v_{2n}(\kappa) y_{2n} \qquad q_{\rm e}(\kappa) = \sqrt{2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n}} v_{2n}(\kappa) x_{2n}\,.$$

Both $p_{\rm e}(\kappa)$ and $q_{\rm e}(\kappa)$ vanish at $\kappa = 0$, but because of the pole in $g(\kappa)$, $\bar{p}_{\rm e}(\kappa)$ and $\bar{q}_{\rm e}(\kappa)$ do *not* vanish at $\kappa = 0$. These new coordinates are responsible for describing unphysical configurations of the string; $\bar{y}_{\rm e}(0)$ describes a configuration where the midpoint of the c ghost is disconnected from $c(\sigma)$, and $\bar{x}_{\rm e}(0)$ describes a configuration where two opposite delta functions in $b(\sigma)$ coincide at the midpoint:

$$b(\sigma) = \lim_{\epsilon\to 0}\left(\delta\left(\sigma - \frac{\pi}{2} + \epsilon\right) - \delta\left(\sigma - \frac{\pi}{2} - \epsilon\right)\right). \tag{4.10}$$

So the situation for the ghost sector is much like the matter sector: in normalizing the non(anti)commutativity parameter to 1 we must either loose physical degrees of freedom or introduce unphysical ones into the theory.

Therefore, the pole in $g(\kappa)$ has physical significance and we need to understand what it implies about the structure of the star product in the ghost sector. To this end it is useful to understand the effect of the derivations $\delta/\delta x_{\rm o}(0), \delta/\delta y_{\rm o}(0)$ on the algebra. Consider,

$$\Psi *_{b_0}\left(\frac{\delta}{\delta x_{\rm o}(0)}\Phi\right)[u]$$

for two arbitrary but well-behaved string fields. We can calculate this using the kernel representation of the Moyal product [12]:

$$\begin{aligned}\Psi *_{b_0}\left(\frac{\delta}{\delta x_{\rm o}(0)}\Phi\right)[u^3] &= \mathcal{N}_{gh}\int [du^1][du^2]\Psi[u^1]\left(\frac{\delta}{\delta x^2_{\rm o}(0)}\Phi[u^2]\right) K\left[u^1, u^2, u^3\right] \\ &= (-1)^{G_\Phi+1}\mathcal{N}_{gh}\int [du^1][du^2]\Psi[u^1]\Phi[u^2]\frac{\delta}{\delta x^2_{\rm o}(0)} K\left[u^1, u^2, u^3\right], \end{aligned} \tag{4.11}$$

where in the last step we integrated by parts and G_Φ denotes the Grassmann parity of Φ. To proceed we need the explicit formula for K:

$$K[u^1, u^2, u^3] = C\exp\left[-\frac{i}{2}\int_{-\infty}^{\infty}\frac{1}{g(\kappa)}\left(x^A_{\rm e}(\kappa)K^{AB}y^B_{\rm o}(\kappa) - x^A_{\rm o}(\kappa)K^{AB}y^B_{\rm e}(\kappa)\right)\right],$$

where C is a constant (which we will not need) and K^{AB} is a 3×3 matrix,

$$K^{AB} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}^{AB}.$$

We find,

$$\frac{\delta}{\delta x^2_{\rm o}(0)}K[u^1, u^2, u^3] = \lim_{\kappa\to 0}\frac{i}{2}\frac{1}{g(\kappa)}K^{2B}y^B_{\rm e}(\kappa)K\left[u^1, u^2, u^3\right].$$

This clearly *vanishes* as a consequence of the pole in $g(\kappa)$ at $\kappa = 0$. The argument similarly goes through for $y_o(\kappa)$. This implies a surprising fact: any string field which is proportional to $\pi_b(\frac{\pi}{2})$ or $\Pi_{c,L}$ has vanishing star product with any other string field:

$$\Psi *_{b_0} \left[\pi_b\left(\frac{\pi}{2}\right)\Phi\right] = 0\,, \qquad \Psi *_{b_0} \left[\Pi_{c,L}\left(\frac{\pi}{2}\right)\Phi\right] = 0\,. \tag{4.12}$$

Therefore the pole at $\kappa = 0$ implies the existence of null elements in the algebra of string fields. By cyclicity of the kernel we also have,

$$\begin{aligned} \left[\pi_b\left(\frac{\pi}{2}\right)\Psi\right] *_{b_0} \Phi &= 0\,, \qquad & \left[\Pi_{c,L}\left(\frac{\pi}{2}\right)\Psi\right] *_{b_0} \Phi &= 0 \\ \pi_b\left(\frac{\pi}{2}\right)[\Psi *_{b_0} \Phi] &= 0\,, \qquad & \Pi_{c,L}\left(\frac{\pi}{2}\right)[\Psi *_{b_0} \Phi] &= 0\,. \end{aligned} \tag{4.13}$$

A very similar structure concerning $c(\frac{\pi}{2})$ and the full star product was discovered by Okuyama in ref. [13, 32]. We have not reproduced his result, of course, since we are studying the reduced star product in Siegel gauge (also introduced in [13]).

The above argument has the advantage of demonstrating clearly the relationship between the pole in $g(\kappa)$ and the existence of null operators, but may appear formal. If one prefers, equations (4.12)–(4.13) can be established directly from the vertex via the identities,

$$\langle V_3^{\rm red}|\pi_b\left(\frac{\pi}{2}\right)^A = \langle V_3^{\rm red}|\Pi_{c,L}^A = 0\,, \tag{4.14}$$

where $A = 1, 2, 3$ is a state space index and $\langle V_3^{\rm red}|$ is the vertex for the reduced star product. To verify these identities, we need to recall some basic facts about the vertex. By definition,

$$\mathrm{bpz}(|\Psi *_{b_0} \Phi\rangle) = \langle V_3^{\rm red}||\Psi\rangle|\Phi\rangle\,.$$

The vertex takes the form:

$$\langle V_3^{\rm red}| = \langle 0|^1\langle 0|^2\langle 0|^3 \exp\left[X_{mn}^{AB} c_m^A b_n^B\right]\,, \tag{4.15}$$

where the constants X are the ghost Neumann coefficients (see, for example, ref. [33]), and $|0\rangle$ is the ghost zero mode independent vacuum, satisfying $b_n|0\rangle = c_n|0\rangle = 0, 1 \le n$. The spectrum of the ghost Neumann coefficients is [12],

$$\tilde{M}_{mn}^{AB} v_n(\kappa) = \mu^{AB}(\kappa) v_m(\kappa) \qquad \tilde{M}_{mn}^{AB} = \frac{(-1)^{m+1}}{\sqrt{m}} X_{mn}^{AB}\sqrt{n}\,, \tag{4.16}$$

where

$$\begin{aligned} \tilde{\mu}^{11}(\kappa) &= \frac{1}{2\cosh\frac{\pi\kappa}{2} - 1} \\ \tilde{\mu}^{12}(\kappa) &= \frac{\cosh\frac{\pi\kappa}{2} + \sinh\frac{\pi\kappa}{2} - 1}{2\cosh\frac{\pi\kappa}{2} - 1} \\ \tilde{\mu}^{21}(\kappa) &= \frac{\cosh\frac{\pi\kappa}{2} - \sinh\frac{\pi\kappa}{2} - 1}{2\cosh\frac{\pi\kappa}{2} - 1}\,. \end{aligned} \tag{4.17}$$

The remaining $\tilde{\mu}^{AB}$s are related to these by cyclicity. We can now evaluate,

$$\begin{aligned}\langle V_3^{\rm red}|\pi_b(\frac{\pi}{2})^B &= -\frac{1}{\pi}\sum_{m,n=1}^{\infty}\langle V_3^{\rm red}|c_m^A(\delta^{AB}\delta_{mn}-X_{mn}^{AB})(-1)^n \\ &= \frac{1}{\sqrt{\pi}}\langle V_3^{\rm red}|\sum_{m=1}^{\infty}(-1)^{m+1}\sqrt{m}c_m^A\int_{-\infty}^{\infty}d\kappa v_m(\kappa)\left[\delta^{AB}-\tilde{\mu}^{AB}(\kappa)\right]\delta(\kappa),\end{aligned} \tag{4.18}$$

where in the last step we transformed to the κ basis. From eq. (4.17) it is clear that the quantity $\delta^{AB}-\tilde{\mu}^{AB}$ has a zero at $\kappa=0$, thus

$$[\delta^{AB}-\tilde{\mu}^{AB}(\kappa)]\delta(\kappa)=0$$

at least in a distributional sense, proving that $\langle V_3^{\rm red}|\pi_b(\frac{\pi}{2})^B=0$ as claimed. An analogous argument follows for $\Pi_{c,L}$. As with the "commutative" nature of P_L, anomalies are possible for string fields which are sufficiently singular at the midpoint, but at least when $\langle V_3^{\rm red}|\pi_b(\frac{\pi}{2})$ is contracted with Fock space states the sum in eq. (4.18) terminates at a finite mode number and indeed $\pi_b(\frac{\pi}{2})$ is null.

It is worth noting that the existence of the null operators $\pi_b(\frac{\pi}{2}),\Pi_{c,L}$ could have been anticipated naively from the overlap properties of the vertex. To see this, consider the overlap conditions for the ghost momenta π_b and π_c:

$$\langle V_3^{\rm red}|\pi_b(\sigma)^A=-\langle V_3^{\rm red}|\pi_b(\pi-\sigma)^{A+1}\qquad\langle V_3^{\rm red}|\pi_c(\sigma)^A=\langle V_3^{\rm red}|\pi_c(\pi-\sigma)^{A+1}\,. \tag{4.19}$$

Supposing that the overlap holds even in the boundary case $\sigma=\frac{\pi}{2}$, we find

$$\langle V_3^{\rm red}|\pi_b\left(\frac{\pi}{2}\right)^1=-\langle V_3^{\rm red}|\pi_b\left(\frac{\pi}{2}\right)^2=\langle V_3^{\rm red}|\pi_b\left(\frac{\pi}{2}\right)^3=-\langle V_3^{\rm red}|\pi_b\left(\frac{\pi}{2}\right)^1 \tag{4.20}$$

thus $\langle V_3^{\rm red}|\pi_b(\frac{\pi}{2})^A=0$. To see that $\Pi_{c,L}$ should annihilate the vertex, we must integrate the π_c overlap condition over a half string:

$$\begin{aligned}\langle V_3^{\rm red}|\Pi_{c,L}^1 &= \langle V_3^{\rm red}|\int_0^{\pi/2}d\sigma\pi_c(\sigma)^1 \\ &= \langle V_3^{\rm red}|\int_{\pi/2}^{\pi}d\sigma\pi_c(\sigma)^2 \\ &= -\langle V_3^{\rm red}|\int_0^{\pi/2}d\sigma\pi_c(\sigma)^2 \\ &= -\langle V_3^{\rm red}|\Pi_{c,L}^2\end{aligned} \tag{4.21}$$

which by cyclicity implies $\langle V_3^{\rm red}|\Pi_{c,L}=0$. Here we used the Siegel gauge condition

$$b_0=\frac{1}{\pi}\int_0^{\pi}d\sigma\pi_c(\sigma)=0 \tag{4.22}$$

which implies,

$$\int_0^{\pi/2}d\sigma\pi_c(\sigma)=-\int_{\pi/2}^{\pi}d\sigma\pi_c(\sigma) \tag{4.23}$$

in particular, $\Pi_{c,L}$ is only a null operator in the Siegel gauge, just like P_L is only a commutative coordinate at zero momentum. If we considered the full star product, $\Pi_{c,L}$ would be replaced by $c(\frac{\pi}{2})$.

The existence of null operators has several important structural consequences in open string field theory. One of the most interesting is that they can be used — with the help of the Batalin-Vilkovisky formalism [35, 34] — to provide a formal proof that the Feynmann diagrams generated by the Witten vertex provide a complete single covering of the moduli space of open Riemann surfaces [36, 37]. For more details we refer the reader to [36], but the essential idea is to decompose the string field into the sum of two terms:

$$|\Psi\rangle = \phi^i \Phi_i + \psi^i \Psi_i \,,$$

where Φ_i, Ψ_i are a basis of states for $\mathcal{H}_{\rm BCFT}$ satisfying

$$b_0 \Phi_i = 0 \qquad c(\frac{\pi}{2})\Psi_i = 0$$

and ϕ^i, ψ^i are component fields (at ghost number ≤ 1) and antifields (at ghost number ≥ 2). The key point is that a field and its antifield are always associated with opposite ghost vacua — that is, one of the pair is associated with a state Ψ_i which annihilates the vertex as a result of the fact that $c(\frac{\pi}{2})$ is null. Thus fields never couple to their antifields in the classical master action, implying that the classical action recieves no quantum corrections. Another important application occurs in the initial value formulation of open string field theory [36], where the existence of null operators is required by gauge invariance and the consistency of the theory's contraints with time evolution.

5. Null operators versus the identity

The existence of null operators brings a paradox: An algebra with null elements cannot have an identity, but there is a well-known string field I which (under "reasonable" circomstances[14]) behaves as an identity for the star product. In the Moyal representation, the identity string field (for the reduced product) is:

$$I[x_{2n-1}, y_{2n-1}, p_{2n}, q_{2n}] = \mathcal{N}_{gh}^{-1} \,.$$

Up to the normalization eq. (4.1) this is just the number 1. Indeed, it seems trivial that 1 is the identity for a Moyal product. However, we have seen that null operators are associated with poles in the nonanticommutativity parameter, and at such poles the interpretation of the star algebra as a Moyal product is misleading. The fact that I behaves like an identity therefore cannot be taken for granted, and for example we should expect subtleties when trying to evaluate the product,

$$I *_{b_0} \left(\pi_b\left(\frac{\pi}{2}\right)\Psi\right). \tag{5.1}$$

Since I is the identity but $\pi_b(\frac{\pi}{2})$ is null, it is not clear whether this expression will evaluate to 0 or $\pi_b(\frac{\pi}{2})\Psi$.

[14] Subtle problems with the identity string field have appeared in many forms down through the years [38, 39, 40], and what we will discuss here is yet another example.

In this section we attempt to evaluate eq. (5.1) and identify the source of an anomaly in the product. In short, the value of the eq. (5.1) depends on the order in which we integrate the string fields involved against the kernel for the Moyal product. However, in this section we will perform our calculations using the three string vertex $\langle V_3^{\rm red}|$ so that we can make use of familiar squeezed state techniques, but the vertex eq. (4.15) and the kernel eq. (4.11) are trivially related:

$$K[u^1,u^2,u^3] = \langle V_3^{\rm red}||u^1\rangle|u^2\rangle|u^3\rangle\,,$$

where $|u\rangle$ are the eigenstates of $u = (\vec{y}(\kappa),\vec{x}(\kappa))$. We will express the vertex and the identity in terms of oscillators in the kappa basis:

$$\begin{aligned}\beta_{\rm o}(\kappa) &= \sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n-1}(\kappa)}{\sqrt{2n-1}}b_{2n-1} &\qquad \beta_{\rm e}(\kappa) &= i\sqrt{2}\sum_{n=1}^{\infty}\frac{v_{2n}(\kappa)}{\sqrt{2n}}b_{2n}\\ \chi_{\rm o}(\kappa) &= \sqrt{2}\textstyle\sum_{n=1}^{\infty}\sqrt{2n-1}v_{2n-1}(\kappa)c_{2n-1} &\qquad \chi_{\rm e}(\kappa) &= i\sqrt{2}\textstyle\sum_{n=1}^{\infty}\sqrt{2n}v_{2n}(\kappa)c_{2n}\end{aligned} \tag{5.2}$$

which satisfy

$$\{\vec{\beta}(\kappa),\vec{\chi}^{+}(\kappa')\} = \mathbb{I}_{2\times 2}\delta(\kappa-\kappa') \qquad \kappa,\kappa'>0\,,$$

where we use vector notation $\vec{x} = (x_{\rm e}, x_{\rm o})$. The vertex and the identity take the form,

$$\begin{aligned}|I\rangle &= C\exp\left[\int_0^\infty d\kappa\vec{\beta}^{+}(\kappa)\cdot\vec{\chi}^{+}(\kappa)\right]|0\rangle\\ \langle V_3^{\rm red}| &= \langle 0|\langle 0|\langle 0|\exp\left[\frac{1}{2}\int_0^\infty d\kappa\vec{\chi}^{A}(\kappa)\cdot M^{AB}(\kappa)\cdot\vec{\beta}^{B}(\kappa)\right],\end{aligned} \tag{5.3}$$

where C is a normalization, $A,B=1,2,3$ are state space indices and $M^{AB}(\kappa)$ is given by,

$$M^{AB}(\kappa) = \begin{pmatrix} 2\tilde{\mu} & \tilde{\mu}^{12}+\tilde{\mu}^{21}-\sigma_y(\tilde{\mu}^{12}-\tilde{\mu}^{21}) & \tilde{\mu}^{12}+\tilde{\mu}^{21}+\sigma_y(\tilde{\mu}^{12}-\tilde{\mu}^{21})\\ \tilde{\mu}^{12}+\tilde{\mu}^{21}+\sigma_y(\tilde{\mu}^{12}-\tilde{\mu}^{21}) & 2\tilde{\mu} & \tilde{\mu}^{12}+\tilde{\mu}^{21}-\sigma_y(\tilde{\mu}^{12}-\tilde{\mu}^{21})\\ \tilde{\mu}^{12}+\tilde{\mu}^{21}-\sigma_y(\tilde{\mu}^{12}-\tilde{\mu}^{21}) & \tilde{\mu}^{12}+\tilde{\mu}^{21}+\sigma_y(\tilde{\mu}^{12}-\tilde{\mu}^{21}) & 2\tilde{\mu}\end{pmatrix}^{AB}, \tag{5.4}$$

where the $\tilde{\mu}$s are given in eq. (4.17). We may write $\pi_b(\frac{\pi}{2})$ in terms of κ oscillators,

$$\pi_b\left(\frac{\pi}{2}\right) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}d\kappa\delta(\kappa)[\vec{\chi}(\kappa)-\vec{\chi}^{+}(\kappa)]\cdot\vec{O}\,,\qquad \vec{x}\cdot\vec{O} = x_{\rm o}\,.$$

For now we will keep the delta function explicit and not evaluate the integral until we know what type of test function appears in the integrand.

In oscillator language, evaluating eq. (5.1) amounts to calculating

$$\langle A| = \langle V_3^{\rm red}|\pi_b\left(\frac{\pi}{2}\right)^{(2)}|I\rangle^{(3)} \in \mathcal{H}^{*}{}_{\rm BCFT}\otimes\mathcal{H}^{*}{}_{\rm BCFT}\,. \tag{5.5}$$

First acting $\pi_b(\frac{\pi}{2})$ on the vertex we find,

$$\langle A| = \langle V_3^{\rm red}|\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}d\kappa\delta(\kappa)\vec{\chi}^{A}(\kappa)\cdot\vec{Q}^{A}(\kappa)|I\rangle^{(3)}$$

where

$$\vec{Q}^A(\kappa) = [\delta^{A2} - \frac{1}{2}M^{A2}(\kappa)] \cdot \vec{O} = -\frac{1}{2\cosh\frac{\pi\kappa}{2} - 1}\begin{pmatrix} \begin{pmatrix} i\sinh\frac{\pi\kappa}{2} \\ \cosh\frac{\pi\kappa}{2} - 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \cosh\frac{\pi\kappa}{2} - 1 \end{pmatrix} \\ \begin{pmatrix} -i\sinh\frac{\pi\kappa}{2} \\ \cosh\frac{\pi\kappa}{2} - 1 \end{pmatrix} \end{pmatrix}^A . \tag{5.6}$$

Note that $\vec{Q}^A(0) = 0$, so if we were to evaluate the delta function at this point we would indeed find that $\pi_b(\frac{\pi}{2})$ is null. However, let us not do this yet and continue to evaluate $\langle A|$. To this end it is useful to add a source term to the vertex, defining,

$$\langle V_3^{\rm red}(J)| = \langle 0|\langle 0|\langle 0| \exp\left[\int_{-\infty}^{\infty} d\kappa \left(\frac{1}{2}\vec{\chi}^A(\kappa)\cdot M^{AB}(\kappa)\cdot\vec{\beta}^B(\kappa) + \vec{\chi}^A(\kappa)\cdot\vec{J}^A(\kappa)\right)\right] . \tag{5.7}$$

Then we can write,

$$\langle A| = -\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} d\kappa\delta(\kappa)\frac{\delta}{\delta\vec{J}^A(\kappa)}\langle V_3^{\rm red}(J)|I\rangle^{(3)}\cdot\vec{Q}^A(\kappa)\Big|_{J=0} . \tag{5.8}$$

The object $\langle V_3^{\rm red}(J)|I\rangle^{(3)}$ can be calculated using well-known squeezed state techniques, for example with the general formula,

$$\langle 0|\exp\left[c_i G_{ij} b_j + \eta_i b_i + c_i\zeta_i\right]\exp\left[c_i^+ F_{ij} b_j^+\right]|0\rangle = \det(1+FG)\exp\left[-\eta\cdot(1+FG)^{-1}\cdot F\cdot\zeta\right] .$$

Doing this, we find

$$\begin{aligned}\langle V_3^{\rm red}(J)|I\rangle^{(3)} &= C\det\left(1-\frac{1}{2}M^{33}\right)\langle 0|\langle 0|\times \\ &\times\exp\left[\int_0^{\infty} d\kappa\vec{\chi}^a(\kappa)\cdot C^{ab}\cdot\vec{\beta}^b(\kappa) + \int_0^{\infty} d\kappa\vec{\chi}^a(\kappa)\cdot\left(R^a(\kappa)\cdot\vec{J}^3(\kappa)+\vec{J}^a(\kappa)\right)\right] ,\end{aligned}$$

where $a, b = 1, 2$ are state space indices and the matrices C^{ab}, R^a are given by,

$$\begin{aligned} C^{ab} &= \frac{1}{2}\left(M^{ab} + M^{a3}\cdot\frac{1}{2-M^{33}}\cdot M^{3b}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{ab} \\ R^a(\kappa) &= M^{a3}\cdot\frac{1}{2-M^{33}} = \frac{1}{2}\begin{pmatrix} 1+\sigma_y\mathcal{P}\coth\frac{\pi\kappa}{4} \\ 1-\sigma_y\mathcal{P}\coth\frac{\pi\kappa}{4} \end{pmatrix}^a , \end{aligned} \tag{5.9}$$

where $\mathcal{P}$ denotes the principal value. Using this to evaluate eq. (5.8) we find,

$$\langle A| = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} d\kappa\delta(\kappa)\langle V_2|\vec{\chi}^a(\kappa)\cdot(\vec{Q}^a(\kappa) + R^a(\kappa)\cdot\vec{Q}^3(\kappa)) ,$$

where fixing $C = \det(1-\frac{1}{2}M^{33})^{-1}$, $\langle V_2| = \langle V_3^{\rm red}(J)|I\rangle|_{J=0}$ is simply the two string vertex. At this point we assume that $\langle A|$ is contracted with well-behaved string fields, so that we can treat $\vec{\chi}^a(\kappa)$ as if it were a smooth function at $\kappa = 0$. With this assumption, the result simplifies to,

$$\langle A| = \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} d\kappa\delta(\kappa)\frac{\sinh\frac{\pi\kappa}{2}}{2\cosh\frac{\pi\kappa}{2}-1}\left(\mathcal{P}\coth\frac{\pi\kappa}{4}\right)\langle V_2|[\chi_{\rm o}^2(\kappa)-\chi_{\rm o}^1(\kappa)]\,.$$

This expression, involving a product of distributions, is ill-defined. However, as in section 2, we can regulate the distributions involved and take limits in different orders to recover different results. Taking the limit for the delta function first corresponds to acting $\pi_b(\frac{\pi}{2})$ on the vertex first, and indeed in this prescription we find $\delta(\kappa)\sinh\frac{\pi\kappa}{2}\cdot$smooth fuction $=0$. Taking the limit for the principal value distribution first, we find

$$\begin{aligned}\langle A| &= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} d\kappa\delta(\kappa)\frac{\sinh\frac{\pi\kappa}{2}\coth\frac{\pi\kappa}{4}}{2\cosh\frac{\pi\kappa}{2}-1}\langle V_2|\left[\chi_{\rm o}^2(\kappa)-\chi_{\rm o}^1(\kappa)\right]\\ &= \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} d\kappa\delta(\kappa)\langle V_2|\left[\chi_{\rm o}^2(\kappa)-\chi_{\rm o}^1(\kappa)\right]\\ &= \langle V_2|\pi_b(\frac{\pi}{2})^{(2)}\end{aligned}\tag{5.10}$$

so in this regularization, $|I\rangle$ behaves like the identity.

The value of the product eq. (5.1) depends on which distribution we take seriously, the principal value or the delta fuction. However, it is worth noting that by taking the principal value distribution seriously, we effectively introduce an unphysical degree of freedom into the theory, in this case $\bar{p}_{\rm e}(0)$ from eq. (4.9), responsible for describing the configuration eq. (4.10) where $b(\sigma)$ has two opposite overlapping delta functions at the midpoint. We can see this by varing $\langle V_3^{\rm red}(J)|I\rangle$ with respect to this degree of freedom, in other words by acting with the conjugate momentum $\bar{y}_{\rm e}(0)=\lim_{\kappa\to 0} y_{\rm e}(\kappa)/g(\kappa)$:

$$\langle V_3^{\rm red}(J)|I\rangle^{(3)}\bar{y}_{\rm e}(0)^2 = \frac{1}{\sqrt{2}}J_{\rm o}^3(0)\langle V_3^{\rm red}(J)|I\rangle\,.$$

This fails to vanish because of the pole in $R^a(\kappa)$. By introducing this unphysical degree of freedom into the computation, we are essentially normalizing the non-anticommutativity parameter to one, so that (consistently) $\pi_b(\frac{\pi}{2})$ is no longer null.

Our troubles with the identity string field is the simplest example of a more general problem reconciling the existence of null operators and the naive relations between n-point vertices. For example, if we wanted to calculate the star product of three string fields, it is convenient to use the four-string vertex $\langle V_4^{\rm red}| \in \bigotimes_{n=1}^4 \mathcal{H}_{\rm BCFT}$,

$${\rm bpz}(|\Phi *_{b_0} \Psi *_{b_0} \Lambda\rangle) = \langle V_4^{\rm red}||\Phi\rangle|\Psi\rangle|\Lambda\rangle\,.$$

The problem is that, based on the overlap conditions, we would not expect $\pi_b(\frac{\pi}{2})$ to annihilate the four string vertex; so when *three* fields are "multiplied" $\pi_b(\frac{\pi}{2})$ is no longer null! This is clearly inconsistent with the assumption that the trilinear operation $\Phi *_{b_0} \Psi *_{b_0} \Lambda$ can be calculated by successivly taking the star product of the fields involved, $\Phi *_{b_0}(\Psi *_{b_0}\Lambda)$ or $(\Phi *_{b_0}\Psi) *_{b_0}\Lambda$. In other words, there must be an anomaly in the relation,

$$\langle V_4^{\rm red}|^{(1256)} = \langle V_3^{\rm red}|^{(123)}\langle V_3^{\rm red}|^{(456)}|V_2\rangle^{(34)}$$

JHEP03(2005)042

analogous to the anomaly in

$$\langle V_2|^{(12)} = \langle V_3^{\rm red}|^{(123)}|I\rangle^{(3)}$$

as we have just seen. This brings additional difficulty to the notion that string fields form an "algebra," in that the higher point products are not quite equivalent to compositions of lower ones.

6. Complex κ

Consider the following pair of ghost coordinates:

$$\begin{aligned} b\left(\frac{\pi}{2}\right) &= i\sqrt{2}\sum_{n=1}^{\infty}(-1)^{n+1}x_{2n-1} \\ c'\left(\frac{\pi}{2}\right) &= \sqrt{2}\sum_{n=1}^{\infty}(2n-1)(-1)^{n+1}y_{2n-1}, \end{aligned} \tag{6.1}$$

Interestingly, they are purely *anticommutative* under the open string product:

$$\left\{b\left(\frac{\pi}{2}\right),\ p_{2n}\right\}_\star = 0 \qquad \left\{c'\left(\frac{\pi}{2}\right),\ q_{2n}\right\}_\star = 0\,.$$

This can be seen intuitively from the fact that the ghost overlap conditions identify the midpoint values of $b(\sigma)$ and $c'(\sigma)$ locally. More explicitly it can be seen from the fact that $\bar{R}$ has a zero mode,

$$\sum_{m=1}^{\infty}\frac{2\sqrt{2}}{\pi}(-1)^{m+1}(2m-1)R_{2m-1,2n} = (2n)^2T_{2n,2m-1}v_{2m-1} = 0$$

as can be seen from eq. (2.7).

However, this presents us with a puzzle. In the continuous basis, the non-anticommutativity parameter $g(\kappa) = \coth\frac{\pi\kappa}{4}$ has no zeros, so it seems impossible that any linear combination of $x_{\rm o}(\kappa)$ and $y_{\rm o}(\kappa)$ could anticommute with *all* $x_{\rm e}(\kappa)$ and $y_{\rm e}(\kappa)$. The unexpected twist here, however, is that $b(\frac{\pi}{2})$ and $c'(\frac{\pi}{2})$ are actually described by Moyal coordinates with an *imaginary argument*, and for imaginary κ, $\coth\frac{\pi\kappa}{4}$ has zeros. To explain this, it is convenient to define[15],

$$X_{\rm o}(\kappa) = N(\kappa)x_{\rm o}(\kappa) \qquad Y_{\rm o}(\kappa) = N(\kappa)y_{\rm o}(\kappa)\,.$$

This scaling is useful since $X_{\rm o}$ and $Y_{\rm o}$ are analytic everywhere on the complex plane, unlike $x_{\rm o}$ and $y_{\rm o}$; in particular, the change of basis eq. (4.4) becomes defined in terms of functions $N(\kappa)v_n(\kappa) = Ч_n(\kappa)/\sqrt{n}$, which as discussed in the appendix are just polynomials. So

[15] We have seen now many times that attempting to "rescale" coordinates is dangerous, since it has the potential to alter the string's degrees of freedom in an unphysical way. However, for real κ this particular rescaling is not a problem since $N(\kappa)$ is a smooth function without any zeros. For complex κ, $N(\kappa)$ has an infinite number of zeros (branch points), but these all cancel against corresponding singularities in the the coordinates. Therefore, we hold the view that the rescaled Moyal coordinates are really the ones we should have been using all along.

write $b(\frac{\pi}{2})$ in this basis:

$$b(\frac{\pi}{2}) = 2i \int_0^\infty d\kappa \left(\sum_{n=1}^\infty (-1)^{n+1}\, \text{Ч}_{2n-1}(\kappa) \right) \frac{X_{\rm o}(\kappa)}{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}\,.$$

We can immediately see that something strange is going on, since the above series doesn't converge. However, the authors of ref. [28] suggested how this divergent sum could be formally defined. One writes,

$$\begin{aligned} b(\frac{\pi}{2}) &= 2i \int_0^\infty d\kappa \frac{1}{2} \sin\left(2\frac{d}{d\kappa}\right) \kappa \left[\sum_{n=1}^\infty \frac{(-1)^{n+1}}{2n-1}\, \text{Ч}_{2n-1}(\kappa) \right] \frac{X_{\rm o}(\kappa)}{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}} \\ &= 2i\sqrt{\pi} \int_0^\infty d\kappa \left[\frac{1}{2} \sin\left(2\frac{d}{d\kappa}\right) \kappa N(\kappa)\delta(\kappa) \right] \frac{X_{\rm o}(\kappa)}{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}\,. \end{aligned} \tag{6.2}$$

The strange object in the brackets is some type of distribution — let us hold off on this a moment, and assume this makes sense. Integrating by parts,

$$b\left(\frac{\pi}{2}\right) = -\frac{i\sqrt{\pi}}{2} \int_{-\infty}^\infty d\kappa \delta(\kappa) N(\kappa) \kappa \Im \left[\frac{X_{\rm o}(\kappa+2i)}{\frac{2}{\kappa+2i}\sinh\frac{\pi(\kappa+2i)}{2}} \right].$$

Cancelling the pole[16] at $\kappa = 0$ gives,

$$b\left(\frac{\pi}{2}\right) = iX_{\rm o}(2i) \qquad c'\left(\frac{\pi}{2}\right) = -Y_{\rm o}(2i)\,. \tag{6.3}$$

An identical calculation follows for $c'(\frac{\pi}{2})$. $X_{\rm o}$ and $Y_{\rm o}$ are evaluated exactly where $g(\kappa) = \coth\frac{\pi\kappa}{4}$ has a zero! Still this is very formal, since when calculating star products using these expressions one encounters delta functions evaluated with an *imaginary* argument. It is not at all clear what this means

The conundrum is caused by the exotic distribution appearing in eq. (6.2). It is not a Schwartz distribution, since for a general C^∞ function the infinite rank differential operator $\sin(2\frac{d}{dk})$ does not converge. The convergence of $\sin(2\frac{d}{dk})$ on a function $f(\kappa)$ is determined by the sum,

$$\sum_{n=0}^\infty \frac{f^{(2n+1)}(\kappa)}{(2n+1)!} 2^{2n+1} < \infty\,. \tag{6.4}$$

This says that a Taylor expansion of f around any particular κ must converge at least within a radius of 2; thinking of f as a function in the complex plane, this means that f must be analytic inside a band outside the real axis extending from $-2 < \Im(\kappa) < 2$. On this restricted space of test functions (much more restricted than C^∞ functions), eq. (6.2) is a perfectly well defined distribution, and our formal calculation has rigorous meaning. To simplify notation, let us introduce the complex delta function distribution [28], defined:

$$\langle \delta_z, f \rangle = \int_{-\infty}^\infty d\kappa \delta(\kappa - z) f(\kappa) \equiv \exp\left(z\frac{d}{d\kappa}\right) f(\kappa)\Big|_{\kappa=0} \qquad z \in \mathbb{C}\,. \tag{6.5}$$

[16] A subtle point: One must be careful not to multiply κ with $\delta(\kappa)$ before cancelling the pole, otherwise the answer would (wrongly) come out to be zero.

f must be analytic inside a band around the real axis $-a < \Im(\kappa) < a$ with $|z| < a$. Rewriting eq. (6.2) we get simply,

$$b\left(\frac{\pi}{2}\right) = i\langle\delta_{2i}, X_{\rm o}\rangle \qquad c'\left(\frac{\pi}{2}\right) = -\langle\delta_{2i}, Y_{\rm o}\rangle\,.$$

This is really just a reexpression of eq. (6.3), but now we have a solid mathematical basis for calculating star products. For example,

$$\begin{aligned}\left\{b\left(\frac{\pi}{2}\right), p_{2n}\right\}_\star &= \frac{1}{2}\int_{-\infty}^{\infty} d\kappa d\kappa' \delta(\kappa - 2i)\frac{Ч_{2n}(\kappa')}{\frac{2}{\kappa'}\sinh\frac{\pi\kappa'}{2}}\{X_{\rm o}(\kappa), Y_{\rm e}(\kappa')\}_\star \\ &= \int_{-\infty}^{\infty} d\kappa\delta(\kappa - 2i)\, Ч_{2n}(\kappa)\coth\frac{\pi\kappa}{4} \\ &= \exp\left(2i\frac{d}{d\kappa}\right) Ч_{2n}(\kappa)\coth\frac{\pi\kappa}{4}\bigg|_{\kappa=0} = Ч_{2n}(2i)\cot\frac{\pi}{2} = 0\,.\end{aligned}$$

This vanishes, as claimed, because the cotangent has a zero. Notice that $Ч_{2n}(\kappa)\coth\frac{\pi\kappa}{4}$ is analytic in the band $-2 \leq \Im(\kappa) \leq 2$ so the calculation is well defined (the pole of the cotangent gets cancelled by the zero of $Ч_{2n}$). An analogous calculation shows that $c'(\frac{\pi}{2})$ is anticommutative. The complex delta function eq. (6.5) is an example of a very general class of distributions studied in the mathematical literature, called "ultradistributions" [41].

For those who are uncomfortable with these formal computations, we can explicitly regulate and show that everything works fine. Consider,

$$b_\omega\left(\frac{\pi}{2}\right) = 2i\int_0^\infty d\kappa\frac{e^{2\omega}\cos\omega\kappa}{4N(\kappa)}x_{\rm o}(\kappa) = i\sqrt{2}\sum_{n=1}^{\infty}\frac{1}{i}(i\tanh\omega)^{2n-1}x_{2n-1} + \mathcal{O}(e^{-\omega})\,. \tag{6.6}$$

We can rewrite this,

$$b_\omega\left(\frac{\pi}{2}\right) = \frac{i}{4}\int_{-\infty}^{\infty} d\kappa\frac{e^{i\omega(\kappa-2i)}}{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}X_{\rm o}(\kappa)\,. \tag{6.7}$$

As $\omega \to \infty$, we in principle recover $b(\frac{\pi}{2})$, but the integrand oscillates with infinite frequency and diverging amplitude. Naively, it is surprising that this limit makes any sense at all, but with the proper restrictions it is perfectly sensible. Consider an almost everywhere analytic test function $f(\kappa)$ which has simple poles at $\kappa_n = a_n + ib_n$ and branch points at $\theta_n = c_n + id_n$. Evaluate the integral

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\kappa\frac{e^{i\omega(\kappa-2i)}}{\kappa - 2i}f(\kappa) \;=\; & f(2i) + \sum_{0<b_n<D} e^{-\omega(b_n-2)}e^{i\omega a_n}\frac{\text{Res}(f,\kappa_n)}{\kappa_n - 2i} + \\ & + \sum_{0<d_m<D}\frac{e^{\omega(2-d_m)}}{2\pi i}\int_{\theta_m} d\kappa\frac{e^{i\omega\Re(\kappa)}}{\kappa - 2i}f(\kappa) + \\ & + \frac{e^{\omega(2-D)}}{2\pi i}\int_C d\kappa\frac{e^{i\omega\Re(\kappa)}}{\kappa - 2i}f(\kappa)\,.\end{aligned}$$

The contour C runs from $-\infty$ to ∞ parallel and a distance $D > 2$ above the real axis. Branch cuts also have been chosen so that they run parallel to the real axis. Taking ω to infinity, depending on the location of the poles and branch points, we can have the following possibilities: If $0 \le b_n < 2$ or $0 \le d_n < 2$ the integral blows up; if $b_n,\ d_n = 2$ the integral oscillates but does not converge to a definite value; if $2 < b_n$ and $2 < d_n$ everything vanishes except $f(2i)$. Therefore:

$$\frac{1}{2\pi i}\lim_{\omega\to\infty}\int_{-\infty}^{\infty} d\kappa \frac{e^{i\omega(\kappa-2i)}}{\kappa-2i} f(\kappa) = \begin{cases} \text{undefined} \quad \text{if } f(\kappa) \text{ has poles or} \\ \text{branch points for } 0 \le \Im(\kappa) \le 2 \; . \\ f(2i) \qquad \text{otherwise} \end{cases} \tag{6.8}$$

The sequence of functions $e^{i\omega(\kappa-2i)}/(\kappa-2i)$ therefore converges (in the sense of distributions) precisely to the complex delta function δ_{2i} discussed earlier.[17] We can use this result directly to calculate the commutator,

$$\{b(\frac{\pi}{2}), p_{2n}\}_\star = \frac{i}{2}\lim_{\omega\to\infty}\int_{-\infty}^{\infty} d\kappa \frac{e^{i\omega(\kappa-2i)}}{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}\, \text{Ч}_{2n}(\kappa)\coth\frac{\pi\kappa}{4}\,,$$

which is seen to vanish as expected.

Since the cotangent has a zero at $\kappa = 2i$, the even Moyal coordinates should also be anticommutative at $\kappa = 2i$. The even coordinates have a simple interpretation in terms of the midpoint ghost momenta:

$$\langle \delta_{2i}, X_{\mathrm{e}}\rangle = \frac{\pi}{i}\pi_c\left(\frac{\pi}{2}\right) \qquad \langle \delta_{2i}, Y_{\mathrm{e}}\rangle = \pi\cdot\pi_b'\left(\frac{\pi}{2}\right).$$

Under the overlap $\pi_c(\frac{\pi}{2})$ and $\pi_b'(\frac{\pi}{2})$ are identified, and so indeed they should (naively) be anticommutative. However, translating to the mixed basis we see that the matrix R does not have a corresponding zero mode. The vector which "should be" the zero mode, $w_{2n} = \sqrt{2}(-1)^{n+1}$, is far outside of the domain of R. How do we understand this? Calculating a typical anticommutator,

$$\left\{y_{2n-1}, \pi_c\left(\frac{\pi}{2}\right)\right\}_\star = \frac{i}{\pi}\frac{2\sqrt{2}}{2n-1}\left\langle \delta_{2i},\, \text{Ч}_{2n-1}(\kappa)\coth\frac{\pi\kappa}{4}\right\rangle = \infty\,. \tag{6.9}$$

This diverges because $\text{Ч}_{2n-1}(\kappa)\coth\frac{\pi\kappa}{2}$ has a pole at $\kappa = 0$, so it is not a valid test function for δ_{2i}. (Note the importance of keeping track of domains — a naive evaluation would indeed give $\text{Ч}_{2n-1}(2i)\coth\frac{\pi\cdot 2i}{2} = 0$). However, the fact that eq. (6.9) diverges does not mean that $\pi_c(\frac{\pi}{2})$ is not anticommutative when multiplied with suitably restricted functionals — any functional of coordinates $\int_0^\infty d\kappa f(\kappa) x_o(\kappa)$ for which f is analytic in the band $-2 \le \Im(\kappa) \le 2$ and vanishes linearly (or faster) at $\kappa = 0$ will (anti)commute with $\pi_c(\frac{\pi}{2})$.[18] In particular, the Fourier modes y_{2n-1} do not satisfy this restriction, and so their anticommutator with $\pi_c(\frac{\pi}{2})$ diverges.

[17] A delicate point: convergence of eq. (6.8) does not strictly require that f be analytic for $0 \le \Im(\kappa) \le 2$. f may have poles order $n \in \mathbb{Z} \ge 2$, and the integral will have the same value. By contrast, convergence of the differential operator $\exp(2i\frac{d}{d\kappa})$ requires analyticity everywhere in $0 \le \Im(\kappa) \le 2$, so eq. (6.8) and eq. (6.5) are not completely equivalent, though they agree whenever both expressions are defined.

[18] By our earlier discussion, functionals of these coordinates are independent of transformations which break the string at its midpoint.

Of course, the cotangent has zeros not just at $2i$, but for every odd multiple of $2i$ — $\kappa = 6i, 10i, \ldots$. Therefore, on a suitably restricted space of functionals, the ghost sector actually has an infinite number of distinct anticommutative coordinates, four for every positive odd multiple of $2i$. Using some identities given in the appendix we find their explicit form to be:

$$
\begin{aligned}
\langle \delta_{2i(2n-1)}, X_{\mathrm{e}} \rangle &= \pi i \frac{(-1)^n}{2n-1} \, Ч_{2n-1}\left(2\frac{d}{d\sigma}\right) \pi_c(\sigma)\Big|_{\sigma=\pi/2} \\
\langle \delta_{2i(2n-1)}, Y_{\mathrm{e}} \rangle &= -\pi \frac{(-1)^n}{2n-1} \, Ч_{2n-1}\left(2\frac{d}{d\sigma}\right) \pi_b'(\sigma)\Big|_{\sigma=\pi/2} \\
\langle \delta_{2i(2n-1)}, X_{\mathrm{o}} \rangle &= i \frac{(-1)^n}{2n-1} \, Ч_{2n-1}\left(2\frac{d}{d\sigma}\right) b(\sigma)\Big|_{\sigma=\pi/2} \\
\langle \delta_{2i(2n-1)}, Y_{\mathrm{o}} \rangle &= \frac{(-1)^n}{2n-1} \, Ч_{2n-1}\left(2\frac{d}{d\sigma}\right) c'(\sigma)\Big|_{\sigma=\pi/2} .
\end{aligned} \tag{6.10}
$$

As the zero gets further from the real axis, the corresponding anticommutative coordinates are given by progressively higher derivatives of the ghost coordinates and momenta at the string midpoint. Each coordinate is naively anticommutative according to the overlap conditions, but is in fact anticommutative only in an increasingly restricted sense as we move further up in the complex plane. This observation also holds in the matter sector, where the tangent has a zero for every even multiple of $2i$. For each zero there is a pair of commutative coordinates, ($X(\kappa) = N(\kappa)x(\kappa)$ and $Y(\kappa) = N(\kappa)y(\kappa)$ from eq. (2.14))

$$
\begin{aligned}
\langle \delta_{4in}, X \rangle &= i \frac{(-1)^n}{2n} \; Ч_{2n}\left(2\frac{d}{d\sigma}\right) x'(\sigma)\Big|_{\sigma=\pi/2} \\
\langle \delta_{4in}, Y \rangle &= -\pi \frac{(-1)^n}{2n} \; Ч_{2n}\left(2\frac{d}{d\sigma}\right) p(\sigma)\Big|_{\sigma=\pi/2} ,
\end{aligned} \tag{6.11}
$$

for $n \geq 1$.

The non(anti)commutativity parameters $\theta(\kappa), g(\kappa)$ also have an infinite sequence of poles in the complex plane, which as we have seen are equally important as zeros. Not surprisingly, variations with respect to the Moyal coordinates at these poles have a simple interpretation in terms of midpoint degrees of freedom. In the ghost sector, we have

$$
\begin{aligned}
\left\langle \delta_{4in}, N \cdot \frac{\delta}{\delta x_{\mathrm{e}}} \right\rangle &= \frac{i(-1)^n}{2n} \, Ч_{2n}\left(2\frac{\partial}{\partial\sigma}\right) c'(\sigma)\Big|_{\sigma=\pi/2} \\
\left\langle \delta_{4in}, N \cdot \frac{\delta}{\delta x_{\mathrm{o}}} \right\rangle &= \frac{i\pi(-1)^{n+1}}{2n} \, Ч_{2n}\left(2\frac{\partial}{\partial\sigma}\right) \pi_b'(\sigma)\Big|_{\sigma=\pi/2} \\
\left\langle \delta_{4in}, N \cdot \frac{\delta}{\delta y_{\mathrm{e}}} \right\rangle &= \frac{(-1)^n}{2n} \, Ч_{2n}\left(2\frac{\partial}{\partial\sigma}\right) b(\sigma)\Big|_{\sigma=\pi/2} \\
\left\langle \delta_{4in}, N \cdot \frac{\delta}{\delta y_{\mathrm{o}}} \right\rangle &= \frac{\pi(-1)^{n+1}}{2n} \, Ч_{2n}\left(2\frac{\partial}{\partial\sigma}\right) \pi_c(\sigma)\Big|_{\sigma=\pi/2}
\end{aligned} \tag{6.12}
$$

while in the matter sector,

$$\begin{aligned}\left\langle \delta_{2i(2n-1)}, N\cdot\frac{\delta}{\delta x}\right\rangle &= \frac{\pi(-1)^n}{2n-1}\,\text{Ч}_{2n-1}\left(2\frac{\partial}{\partial\sigma}\right)p(\sigma)\Big|_{\sigma=\pi/2}\\ \left\langle \delta_{2i(2n-1)}, N\cdot\frac{\delta}{\delta y}\right\rangle &= \frac{i(-1)^n}{2n-1}\,\text{Ч}_{2n-1}\left(2\frac{\partial}{\partial\sigma}\right)x'(\sigma)\Big|_{\sigma=\pi/2}\end{aligned}\tag{6.13}$$

for $n\geq 1$. All of these operators, when acting on a string field, tend to make star products vanish. As before, when we consider poles increasingly far up in the complex plane, we encounter sequentially higher derivatives of string coordinates at the midpoint, and the operators can be considered null only in an increasingly restricted sense.

As a word of admonition, we should realize that when dealing with with complex Moyal coordinates, ambiguities can creep in which are easy to miss. To see an example, consider the coordinate,

$$\begin{aligned}A &= -\frac{i\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n}p_{2n}\\ &= \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa N(\kappa)}\left[\frac{1}{2}\psi\left(\frac{1}{2}+\frac{i\kappa}{4}\right)+\frac{1}{2}\psi\left(\frac{1}{2}-\frac{i\kappa}{4}\right)-\psi\left(\frac{1}{2}\right)\right]y_{\rm e}(\kappa)\,,\end{aligned}\tag{6.14}$$

where $\psi(z)=\frac{d}{dz}\ln\Gamma(z)$ is the digamma function. Up to a nonsingular term, this coordinate is related to the difference between the b ghost momentum on the two halves of the string:

$$\begin{aligned}\Pi_{b,L}-\Pi_{b,R} &= \int_0^{\pi/2}d\sigma\pi_b(\sigma)-\int_{\pi/2}^{\pi}d\sigma\pi_b(\sigma)\\ &= 2A-\frac{2i\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{p_{2n}}{2n}\,.\end{aligned}\tag{6.15}$$

The anticommutator of A with $b(\frac{\pi}{2})$ is ambiguous, as is easy to see in the mixed basis:

$$\left\{b\left(\frac{\pi}{2}\right),\ A\right\}_\star=\begin{cases}\langle w|T|v\rangle=0\\ \langle v|\bar{T}|w\rangle=1\end{cases}.\tag{6.16}$$

This ambiguity is analogous to that of $[\bar{x},P_L]_\star$ discussed earlier. The origin of the ambiguity comes from the fact that A depends on an "unphysical" Moyal coordinate at $2i$,

$$\lim_{\kappa\to 2i}\frac{y_{\rm e}(\kappa)}{N(\kappa)(\kappa-2i)}\tag{6.17}$$

with which $b(\frac{\pi}{2})$ naively does not commute. In particular we can show,

$$\lim_{\kappa\to 2i}(\kappa-2i)N(\kappa)\frac{\delta A}{\delta y_{\rm e}(\kappa)}=1\,.$$

This limit is nonzero because the integrand has a pole at $2i$. It turns out that the coordinate eq. (6.17) is responsible for describing a configuration where the b ghost momentum has two opposite overlapping delta functions at the midpoint:

$$\pi_b(\sigma)=\lim_{\epsilon\to 0}\left[\delta\left(\sigma-\frac{\pi}{2}+\epsilon\right)-\delta\left(\sigma-\frac{\pi}{2}-\epsilon\right)\right]\quad\longrightarrow\quad\frac{y_{\rm e}(\kappa)}{N(\kappa)(\kappa-2i)}=\delta(\kappa-2i)\,.$$

JHEP03(2005)042

Indeed it is strightforward to see that $\Pi_{b,L} - \Pi_{b,R}$ will depend on this configuration, since it adds b momentum to the midpoint of the left half of the string while subtracting it at the midpoint from the right half.

To see the ambiguity of $[A, b(\frac{\pi}{2})]$ in the κ basis we must regulate the integrand,

$$\begin{aligned}
A_\omega &= \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{d\kappa}{\kappa N(\kappa)}\left[\frac{1}{2}\psi\left(\frac{1}{2}+\frac{i\kappa}{4}\right)+\frac{1}{2}\psi\left(\frac{1}{2}-\frac{i\kappa}{4}\right)-\psi\left(\frac{1}{2}\right)-\right.\\
&\qquad\left.-\,2e^{-2\omega}\frac{1+(\frac{\kappa}{2})^2-\cos\omega\kappa+\frac{\kappa}{2}\sin\omega\kappa}{1+(\frac{\kappa}{2})^2}\right]y_{\mathrm{e}}(\kappa)\\
&= \frac{i\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{(i\tanh\omega)^{2n}}{2n}p_{2n}+\mathcal{O}(e^{-4\omega})\,.
\end{aligned}$$

The curious thing about this regulated formula is that the integrand seems to converge to a function as $\omega\to\infty$. This however is only true if we think of the integrand as a function on the real axis; in the complex plane the integrand converges to a "complex" principal value distribution — the oscillatory term serves to regulate the pole at $2i$. We are lead to consider the integral,

$$\begin{aligned}
\left\{b_\omega\left(\frac{\pi}{2}\right),A_{\omega'}\right\}_\star &= \frac{1}{2\pi}\int_{-\infty}^{\infty}d\kappa\frac{\coth\frac{\pi\kappa}{4}}{\sinh\frac{\pi\kappa}{2}}e^{i\omega(\kappa-2i)}\left[\frac{1}{2}\psi\left(\frac{1}{2}+\frac{i\kappa}{4}\right)+\frac{1}{2}\psi\left(\frac{1}{2}-\frac{i\kappa}{4}\right)-\psi\left(\frac{1}{2}\right)-\right.\\
&\qquad\left.-\,2e^{-2\omega'}\frac{1+(\frac{\kappa}{2})^2-\cos\omega'\kappa+\frac{\kappa}{2}\sin\omega'\kappa}{1+(\frac{\kappa}{2})^2}\right].
\end{aligned}$$

Taking $\omega'\to\infty$ first, we must evaluate ,

$$\left\{b_\omega\left(\frac{\pi}{2}\right),A\right\}_\star = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\kappa\frac{e^{i\omega(\kappa-2i)}}{\sinh\frac{\pi\kappa}{2}}\left[\psi\left(\frac{1}{2}-\frac{i\kappa}{4}\right)-\psi\left(\frac{1}{2}\right)+\frac{\pi i}{4}\tanh\frac{\pi\kappa}{4}\right]\coth\frac{\pi\kappa}{4}\,.$$

Closing the contour in the upper half plane, and keeping in mind eq. (6.8), we see a potential problem with convergence from the double pole in $\frac{\coth\frac{\pi\kappa}{4}}{\sinh\frac{\pi\kappa}{2}}$ at $\kappa=0$. However, this is cancelled against a second order zero in $\psi(\frac{1}{2}+\frac{i\kappa}{4})+\psi(\frac{1}{2}-\frac{i\kappa}{4})-\psi(\frac{1}{2})$. Furthermore, the zero of the cotangent cancels the pole in the tangent at $2i$, giving a nonzero contribution to the $\omega\to\infty$ limit,

$$\lim_{\omega\to\infty}\lim_{\omega'\to\infty}\left\{b_\omega\left(\frac{\pi}{2}\right),A_{\omega'}\right\}_\star = \frac{1}{2\pi}(2\pi i)\left(-\frac{1}{2}\right)(2i)=1\,,$$

in agreement with eq. (6.16). Let's now evaluate the limits in the opposite order:

$$\begin{aligned}
&\lim_{\omega'\to\infty}\lim_{\omega\to\infty}\left\{b_\omega\left(\frac{\pi}{2}\right),A_{\omega'}\right\}_\star =\\
&\quad = 1-\frac{1}{\pi}\lim_{\omega'\to\infty}\lim_{\omega\to\infty}\int_{-\infty}^{\infty}d\kappa\frac{e^{i\omega(\kappa-2i)}}{\sinh\frac{\pi\kappa}{2}}e^{-2\omega'}\left[1-\frac{\cos\omega'\kappa-\frac{\kappa}{2}\sin\omega'\kappa}{1+(\frac{\kappa}{2})^2}\right]\coth\frac{\pi\kappa}{4}\,.
\end{aligned}$$

The integrand has a pole at $2i$ which cancels the zero of the cotangent:

$$\lim_{\omega'\to\infty}\lim_{\omega\to\infty}\left\{b_\omega\left(\frac{\pi}{2}\right),A_{\omega'}\right\}_\star = 1-\frac{1}{\pi}\lim_{\omega'\to\infty}2\pi i\left[e^{-2\omega'}\left(-\frac{1}{2}\right)(-1)\frac{1}{i}(\cos 2i\omega'-i\sin 2i\omega')\right]$$

$$= 1 - \lim_{\omega' \to \infty} e^{-2\omega'}(\cosh 2\omega' + \sinh 2\omega') = 0\,. \qquad (6.18)$$

The answer is again in agreement with eq. (6.16).

The results of this section makes it clear that the κ basis in the complex plane contains an infinite sequence of "mirror images" of the Moyal product for real κ, all of these images describing a Moyal products corresponding to half string overlaps of ever higher worldsheet derivatives of $x(\sigma)$ and $p(\sigma)$. The appearance of this structure is interesting and surprising, but due to the highly constrained convergence properties of the complex commutative and null coordinates, their physical applications may seem limited. However, as noted in ref. [28] the kinetic operator for string field theory inevitably involves the use of complex κ, so the structure of the star product in the complex plane must inevitably be important if we want to find nontrivial solutions solutions to the string field equations.

7. Conclusion

In this paper we studied the role of the midpoint in the open string star algebra, finding that the midpoint structure is characterized by the behavior of certain midpoint degrees of freedom under star multiplication. In addition to the known commutative coordinate P_L for the matter sector, we discovered a pair of midpoint coordinates for the ghost sector which *annihilate* the star product, and an infinite number of new coordinates which, in a distributional sense, serve as commutative and null coordinates for the star product. The existence of these degrees of freedom and their properties contradicts the straightforward interpretation of the star product as a matrix product, and can be thought of as a consequence of the fact that the string is a continuous object which cannot be broken into "halves." In fact, we have seen with the examples of $[\bar{x}, P_L]_\star$ and $I *_{b_0} \pi_b(\frac{\pi}{2})\Psi$ that these midpoint properties only tend to be anomalous when we invoke naively unphysical half-string degrees of freedom which cannot be represented with full string Fourier modes.

These midpoint properties of the star product may have several useful applications. As mentioned earlier, they are necessary for ensuring the consistency and gauge invariance of the recently proposed inintial value formulation of open string field theory [36], and can be used to provide a field-theoretic proof that the Feynmann diagrams of open string field theory provide a complete single covering of the moduli space of open Riemann surfaces [36, 37]. In addition, our results should have implications for singular features of star algebra projectors [11, 42], and for the construction of formal solutions based on the identity string field [43]. Also, null operators imply interesting linear constraints on string field theory solutions, for example

$$\pi_b\left(\frac{\pi}{2}\right) Q_B \Psi = 0\,.$$

It would be interesting to see how well this formula converges for the closed string vacuum solution in the level truncation scheme.

There are many directions beyond the current work which are worth exploring. One of the most interesting is the extension of our results beyond Siegel gauge and zero momentum. The naive expectation is that including zero modes will change the commutative and null coordinates, replacing P_L with $x(\frac{\pi}{2})$ and $\Pi_{c,L}$ with $c(\frac{\pi}{2})$. However, there are reasons to believe that the story is more subtle than this — for example, the work of ref. [36, 44] suggests that there is a fundamental obstruction to expressing string fields as a local function of $x(\frac{\pi}{2})$, so a completely local formulation of the star product is probably impossible. These issues have profound implications for the locality properties of string field theory and its proposed initial value formulation [36], so a more detailed understanding seems crucial. Beyond this, it might be useful to investigate singular properties of the star product for the open superstring [45] or for the string in the presence of Kalb-Ramond flux [46]. Another issue is the proof of gauge invariance within the Moyal formulation of string field theory. Such a study could bring useful insight into the subtle inner workings and consistency of the theory, and perhaps allow us to get a handle on the space of fields for which we can expect the action to be gauge invariant. The eventual hope is to achieve a complete operational understanding of the star product and a useful characterization of the algebra of string fields. Hopefully the work of this paper is an important step in this direction.

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A. More about the v_n's

The harmonic oscillator hamiltonian is

$$H = a^+a + \frac{1}{2}\,.$$

This operator has a discrete spectrum of eigenvectors $|n\rangle$ (in our conventions $1 \le n < \infty$) corresponding to the $n-1$st excited state of the harmonic oscillator. On the other hand, the position operator,

$$X = \frac{i}{\sqrt{2}}(a - a^+)$$

has a continuous spectrum of eigenvectors, $|x\rangle$ $(-\infty < x < \infty)$. In quantum mechanics we are often interested in the inner product,

$$\phi_n(x) = \langle x|n-1\rangle = H_n(x)\frac{e^{-x^2/2}}{(\pi 4^n (n!)^2)^{1/4}}\,.$$

The ϕ_n's are of course the familiar Hermite functions and the H_n's are the Hermite polynomials. But in this paper we do not care so much about X and its eigenvalues. More interesting is the operator [23],

$$\mathrm{K}_1 = -\sqrt{aa^+}a^+ - a\sqrt{aa^+}\,.$$

K_1 has a continuous spectrum of eigenvectors conventionally labelled by κ for $-\infty < \kappa < \infty$. Consider wavefunctions formed from the inner product,

$$v_n(\kappa) = \langle \kappa | n \rangle = \frac{\text{Ч}_n(\kappa)}{\sqrt{n}\sqrt{\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}}}\,.$$

The functions Ч_n are simply a set of polynomials, like the Hermite polynomials, but orthogonal with respect to the weight function [32],

$$w(\kappa) = \left(\frac{2}{\kappa}\sinh\frac{\pi\kappa}{2}\right)^{-1}\,.$$

The Ч_n's are unfortunately not one of the classical orthogonal polynomials we would easily find in a handbook. Therefore it's worthwhile cataloging a few of their properties:

Differential equation: the statement that the v_n's are eigenstates of H translates to the differential equation [28, 29]:

$$\sin\left(2\frac{d}{d\kappa}\right)\kappa\,\text{Ч}_n(\kappa) = 2n\,\text{Ч}_n(\kappa)\,.$$

Generating function [23]:

$$\sum_{n=1}^{\infty}\frac{z^n}{n}\,\text{Ч}_n(\kappa) = \frac{1}{\kappa}\left(1 - e^{-\kappa\tan^{-1}z}\right) = f_\kappa(z)\,.$$

Recurrence relations: the polynomials can be systematically generated from the relation

$$\text{Ч}_{n+1}(\kappa) + \text{Ч}_{n-1}(\kappa) = -\frac{\kappa}{n}\,\text{Ч}_n(\kappa)\,;$$

after defining $\text{Ч}_0(\kappa) = 0$ and $\text{Ч}_1(k) = 1$. The first few polynomials are explicitly:

$$\begin{aligned}
\text{Ч}_1(\kappa) &= 1 & \text{Ч}_4(\kappa) &= -\frac{1}{6}\kappa^3 + \frac{4}{3}\kappa \\
\text{Ч}_2(\kappa) &= -\kappa & \text{Ч}_5(\kappa) &= \frac{1}{24}\kappa^4 - \frac{5}{6}\kappa^2 + 1 \\
\text{Ч}_3(\kappa) &= \frac{1}{2}\kappa^2 - 1 & \text{Ч}_6(\kappa) &= -\frac{1}{120}\kappa^5 + \frac{1}{3}\kappa^3 - \frac{23}{15}\kappa\,.
\end{aligned}$$

A graph of the first five $v_n(\kappa)$'s is figure 3.

Ladder operators:

$$\begin{aligned}
\cos^2\left(\frac{d}{d\kappa}\right)\kappa\,\text{Ч}_n(\kappa) &= -n\,\text{Ч}_{n+1}(\kappa)\,, \\
\sin^2\left(\frac{d}{d\kappa}\right)\kappa\,\text{Ч}_n(\kappa) &= -n\,\text{Ч}_{n-1}(\kappa)\,.
\end{aligned}$$

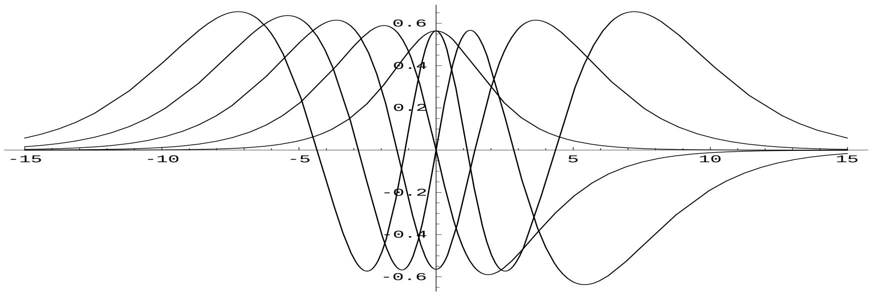

Figure 3: The first five v_ns.

Special values:

$$
\begin{aligned}
&\text{Ч}_{2n}(0) = 0 \qquad \text{Ч}_{2n-1}(0) = (-1)^{n+1} \\
&\text{Ч}'_{2n}(0) = (-1)^n \sum_{m=1}^{n} \frac{1}{2m-1} \qquad \text{Ч}'_{2n-1}(0) = 0 \\
&\text{Ч}_n(2i) = n(-i)^{n-1} \,.
\end{aligned}
$$

Exotic Theorem:

$$
m\ i^m\ \text{Ч}_n(2im) = -n\ i^{-n}\ \text{Ч}_m(-2in)\,.
$$

Derivative:

$$
\begin{aligned}
\text{Ч}'_{2n}(\kappa) &= -\frac{\text{Ч}_{2n}(\kappa)}{\kappa} + 2n \sum_{m=1}^{n} \frac{(-1)^{m-n+1}}{2n-(2m-1)} \frac{\text{Ч}_{2m-1}(\kappa)}{2m-1} \\
\text{Ч}'_{2n-1}(\kappa) &= \frac{(-1)^{n+1} - \text{Ч}_{2n-1}(\kappa)}{\kappa} + (2n-1) \sum_{m=1}^{n-1} \frac{(-1)^{m-n}}{(2n-1)-2m} \frac{\text{Ч}_{2m}(\kappa)}{2m} \,.
\end{aligned}
$$

Integral representations [28, 30]:

$$
\begin{aligned}
\text{Ч}_n(\kappa) &= \frac{n}{2\pi} i^{n-1} w(\kappa) \int_{-\infty}^{\infty} du\ e^{i\kappa u} \frac{\tanh^{n-1} u}{\cosh^2 u} \\
&= \frac{n}{2\pi} i^{n-1} w(\kappa) \left(\int_1^{\infty} + \int_{-\infty}^{-1} \right) dx\ \frac{e^{i\kappa \coth^{-1} x}}{x^{n+1}} \\
&= \frac{n}{2\pi i} \oint_{z=0} dz\ \frac{f_\kappa(z)}{z^{n+1}} \,.
\end{aligned}
\tag{A.1}
$$

Darboux-Christoffel formula:

$$
\sum_{n=1}^{N} \frac{1}{n} \text{Ч}_n(\kappa)\, \text{Ч}_n(\kappa') = -\frac{\text{Ч}_{N+1}(\kappa)\, \text{Ч}_N(\kappa') - \text{Ч}_{N+1}(\kappa')\, \text{Ч}_N(\kappa)}{\kappa - \kappa'} \,.
$$

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